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Towards the quantum S-matrix of the Pohlmeyer reduced version of $AdS_5 \times S^5$ superstring theory

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Abstract

We investigate the structure of the quantum S-matrix for perturbative excitations of the Pohlmeyer reduced version of the $AdS_5 \times S^5$ superstring following arXiv:0912.2958. The reduced theory is a fermionic extension of a gauged WZW model with an integrable potential. We use as an input the result of the one-loop perturbative scattering amplitude computation and an analogy with simpler reduced $AdS_n \times S^n$ theories with $n = 2, 3$. The reduced $AdS_2 \times S^2$ theory is equivalent to the $\mathcal{N} = 2$ 2-d supersymmetric sine-Gordon model for which the exact quantum S-matrix is known. In the reduced $AdS_3 \times S^3$ case the one-loop perturbative S-matrix, improved by a contribution of a local counterterm, satisfies the group factorization property and the Yang-Baxter equation, and reveals the existence of a novel quantum-deformed 2-d supersymmetry which is not manifest in the action. The one-loop perturbative S-matrix of the reduced $AdS_5 \times S^5$ theory has the group factorisation property but does not satisfy the Yang-Baxter equation suggesting some subtlety with the realisation of quantum integrability. As a possible resolution, we propose that the S-matrix of this theory may be identified with the quantum-deformed $[\mathfrak{psu}(2|2)]^2 \ltimes \mathbb{R}^2$ symmetric R-matrix constructed in arXiv:1002.1097. We conjecture the exact all-order form of this S-matrix and discuss its possible relation to the perturbative S-matrix defined by the path integral. As in the $AdS_3 \times S^3$ case the symmetry of the S-matrix may be interpreted as an extended quantum-deformed 2-d supersymmetry.

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Contents

1 Introduction

The aim of this paper is to continue the investigation [1, 2] into the S-matrix of the Pohlmeyer reduced version of superstring theory on $AdS_5 \times S^5$. One motivation is to shed light on an eventual first-principles solution of the $AdS_5 \times S^5$ superstring based on quantum integrability.

We shall view the reduced $AdS_5 \times S^5$ theory as a member of a class of $AdS_n \times S^n$ ($n = 2, 3, 5$) theories which are Pohlmeyer reductions of Green-Schwarz superstring sigma models based on $AdS_n \times S^n$ supercosets. These reduced theories [3, 4, 5] are fermionic extensions of generalised sine-Gordon models. Various examples of such bosonic models (called also "symmetric space sine-Gordon models") based on a G/H gauged WZW theory with an integrable potential term were considered in, e.g., [6, 7, 8, 9, 10, 11]. Due to their relation via Pohlmeyer reduction to classical GS superstring theory on $AdS_n \times S^n$ (and, more generally, of their bosonic truncations to classical string theory on symmetric spaces) there has been recent interest in these models [12, 13, 14, 15, 16, 17, 18].

Let us briefly recall the basic setup. The fields of the reduced theory are all valued in certain subspaces of a particular representation of the superalgebra $\hat{\mathfrak{f}}$, whose corresponding supergroup $\hat{F}$ is the global symmetry of the original superstring sigma model. The latter is based on the supercoset $\hat{F}/G$, where G is a bosonic subgroup of $\hat{F}$ (fixed by a $\mathbb{Z}_4$) and is the gauge group of the superstring sigma model. For the $AdS_5 \times S^5$ superstring [19] the supercoset is

$$\frac{PSU(2,2|4)}{Sp(2,2) \times Sp(4)} .$$

The gauge group H of the reduced theory is a subgroup of G that appears upon solving the Virasoro constraints. The reduced theory action is a fermionic extension of the G/H gauged WZW theory with an integrable potential [3] [1]

$$\begin{aligned}\mathcal{S} = \frac{k}{8\pi\nu}\mathrm{STr}\Big[&\frac{1}{2}\int d^2x\ \ g^{-1}\partial_+ g\ g^{-1}\partial_- g\ -\frac{1}{3}\int d^3x\ \ \epsilon^{mnl}\ g^{-1}\partial_m g\ g^{-1}\partial_n g\ g^{-1}\partial_l g \\ &+ \int d^2x\ \Big(A_+\partial_- g g^{-1} - A_- g^{-1}\partial_+ g - g^{-1}A_+ g A_- + \tau(A_+)A_- + \mu^2\,(g^{-1}TgT - T^2)\Big) \\ &+ \int d^2x\ (\Psi_{_L} T D_+ \Psi_{_L} + \Psi_{_R} T D_- \Psi_{_R} + \mu\, g^{-1}\Psi_{_L} g \Psi_{_R})\Big] .\end{aligned} \tag{1.1}$$

Here $g \in G$, $A_\pm \in \mathfrak{h}$=alg(H) and the fermionic fields $\Psi_{_L}$, $\Psi_{_R}$ take values in fermionic subspaces of $\hat{\mathfrak{f}}$. k is a coupling constant (level) and ν is the index of the representation of G in which g is taken as a matrix. [2] μ

[1] We choose Minkowski signature in 2 dimensions with $d^2x = dx^0dx^1$, $\partial_\pm \equiv \partial_0 \pm \partial_1$. For algebras $[\mathfrak{a}]^2 = \mathfrak{a} \oplus \mathfrak{a}$, i.e. the direct sum. We also use the notation $\Subset$ for a semi-direct sum of algebras and $\ltimes$ for a central extension. For example, for the semi-direct sum $\mathfrak{a} \Subset \mathfrak{b}$ we have the commutation relations: $[\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{a}$, $[\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{b}$ and $[\mathfrak{a}, \mathfrak{b}] \subset \mathfrak{b}$.

[2] In all the theories we consider in the main body of this paper the fundamental representations of SU and Sp groups are used. Therefore, we have $\nu = \frac{1}{2}$. In particular, as in [5], the reduced $AdS_3 \times S^3$ theory is written in terms of a single copy of the fundamental representation of the supergroup $PSU(1,1|2)$. In appendix G bosonic theories with $G = SO(N+1)$ are considered. For these theories with the fundamental representation one has $\nu = 1$.

is a parameter defining the mass of perturbative excitations near $g = \mathbf{1}$. The standard symmetric gauging corresponds to $\tau = \mathbf{1}$ (τ is an automorphism of $\mathfrak{h}$); for an abelian gauge group H there is an option of axial gauging corresponding to $\tau(h) = -h$. The constant matrix T defining the potential commutes with H (see, e.g., [3, 13] for details).

In the case of the $AdS_5 \times S^5$ superstring the Pohlmeyer reduced theory has certain unique features; in particular, it is UV-finite [20] and its one-loop semiclassical partition function is equivalent to that of the original $AdS_5 \times S^5$ superstring [21]. This suggests [3] that it may be quantum-equivalent to the $AdS_5 \times S^5$ superstring. If this were the case, the Pohlmeyer reduced theory could be used as a starting point for a 2-d Lorentz covariant "first-principles" solution of the $AdS_5 \times S^5$ superstring. The Lorentz invariance of (1.1) is a desirable feature as lack of 2-d Lorentz symmetry in the light-cone gauge $AdS_5 \times S^5$ superstring S-matrix leads, e.g., to a complicated structure for the corresponding thermodynamic Bethe Ansatz for the full quantum superstring spectrum (see, e.g., [22] and references therein).

The form of the light-cone gauge $AdS_5 \times S^5$ superstring S-matrix (corresponding to the spin-chain magnon S-matrix on the gauge theory side [23]) is fixed, up to a phase, by the residual global $PSU(2|2) \times PSU(2|2)$ symmetry of the light-cone gauge Hamiltonian [24, 25]. This S-matrix is the starting point for the conjectured Bethe Ansatz solution for the superstring energy spectrum based on its integrability (see [26]). Just as for the standard 2-d sigma models [27] or other similar massive theories [28, 14, 29] the starting point for solving the Pohlmeyer reduced theory is to find its exact S-matrix. Any proposal for the exact quantum S-matrix should be, of course, consistent with the perturbative S-matrix computed from the path integral defined by the classical action.

This motivates the study of the perturbative S-matrix of the Pohlmeyer reduced $AdS_n \times S^n$ superstring. In [1] we computed the tree-level two-particle S-matrix for the $8+8$ massive excitations of the reduced $AdS_5 \times S^5$ theory employing the light-cone $A_+ = 0$ gauge. [3]

Remarkably, the resulting S-matrix factorises in the same way as the non-Lorentz invariant $[\mathfrak{psu}(2|2)]^2 \ltimes \mathbb{R}^3$ symmetric light-cone gauge S-matrix [34, 35, 26] of the $AdS_5 \times S^5$ superstring. The factorised S-matrix has an intriguing similarity with a particular limit of the quantum-deformed $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ invariant R-matrix of [36, 37].

In [2] the perturbative computation was extended to the one-loop level for the bosonic part of the $G/H = SO(N+1)/SO(N)$ theories defined by (1.1). It was argued that the Lagrangian describing the physical fields constructed from the gauged WZW model (1.1) should be supplemented by a particular one-loop counterterm coming from the path integral. For the $G/H = SU(2)/U(1)$ theory [4] the one-loop counterterm contributions to the S-matrix were computed in three different ways, all giving the same result. These contributions were precisely those needed to restore the validity of the Yang-Baxter equation (YBE) at the one-loop level [38, 39, 40] and to match the exact quantum soliton S-matrix proposed in [28].

An alternative approach to exploring these models is based on constructing soliton solutions [15, 17] and semi-classically quantising them. Assuming quantum integrability [14] one can then conjecture an exact soliton S-matrix.

In this paper we investigate the quantum S-matrices for the perturbative massive excitations of the models (1.1) which are the Pohlmeyer reductions of the $AdS_2 \times S^2$, $AdS_3 \times S^3$ and $AdS_5 \times S^5$ superstring models. [5] Studying the three models together turns out to be useful as it reveals certain universal features of their symmetries and S-matrices and thus helps to shed light on the structure of the most non-trivial case of the $AdS_5 \times S^5$ theory.

The reduced $AdS_2 \times S^2$ theory is equivalent [3] to the $\mathcal{N} = 2$ supersymmetric sine-Gordon model [43, 44]. Below we will review the construction of the exact S-matrix for its perturbative excitations [43, 44, 45, 46] and demonstrate the agreement with the direct one-loop computation of scattering amplitudes.

[3]The generalised sine-Gordon models have been mostly studied in the case of abelian gauge groups [30, 31, 32, 33], for which there is an option of axial gauging. In this case the vacuum is unique up to gauge transformations. In the case of a non-abelian H there is a non-trivial vacuum moduli space, no global symmetry and upon integrating out the gauge fields $A_\pm$ one is left with a Lagrangian that has no perturbative expansion about the trivial vacuum. This problem is an artifact of the gauge fixing procedure on g. If instead one chooses the light-cone gauge $A_+ = 0$ [1] one is able to construct a perturbative Lagrangian for the asymptotic excitations and compute the S-matrix. In the case of a non-abelian H [2] this procedure leads to an S-matrix that does not satisfy YBE (already at the tree level).

[4]This theory is classically equivalent to the complex sine-Gordon theory, as seen by fixing a gauge on the group field g and integrating out $A_\pm$.

[5]In this paper by the $AdS_3 \times S^3$ and $AdS_2 \times S^2$ superstring theories we mean the formal supercoset truncations of the full 10-d superstring theories in $AdS_3 \times S^3 \times T^4$ [41] and in $AdS_2 \times S^2 \times T^6$ [42].

In the reduced $AdS_3 \times S^3$ theory the one-loop S-matrix computed starting from the classical Lagrangian does not satisfy the Yang-Baxter equation, but we will show that one can find a local counterterm that restores the YBE and thus integrability, much like in the bosonic complex sine-Gordon theory case discussed in [2]. The addition of the counterterm not only restores the validity of the YBE but also ensures the group factorisation property and leads to a novel *quantum-deformed supersymmetry* of the S-matrix. The existence of a hidden 2-d supersymmetry in the classical reduced $AdS_3 \times S^3$ and $AdS_5 \times S^5$ theories was conjectured, by analogy with the $AdS_2 \times S^2$ case, in [3] and was recently discussed in [16] and demonstrated, at least on-shell, in [47] and also off-shell for the Lagrangian (1.1) in [48].

Assuming that the group-factorisation and the quantum-deformed supersymmetry are true symmetries of the theory we conjecture an exact 2-d Lorentz invariant quantum S-matrix for the perturbative excitations of the reduced $AdS_3 \times S^3$ theory. The phase factor is fixed by the unitarity and crossing constraints (and is similar to that in the reduced $AdS_2 \times S^2$ theory). We check that the resulting exact S-matrix expanded in $1/k$ agrees with our one-loop computation.

In the $AdS_5 \times S^5$ case we observe that the one-loop S-matrix group-factorises in the same way as at the tree level in [1]. However, there is a tree-level anomaly in the YBE [1, 2] which is a general feature of the models (1.1) with a non-abelian gauge group H. As in the bosonic case [5], this anomaly cannot be cancelled by adding a local two-derivative counterterm without breaking the manifest non-abelian symmetry, indicating some subtlety with a realisation of integrability.

Motivated by the quantum-deformed supersymmetry we discovered in the S-matrix of the reduced $AdS_3 \times S^3$ theory and the quantum-deformed non-abelian symmetry expected in soliton S-matrices of the bosonic theories [14, 15, 17] here we conjecture that the S-matrix for the perturbative massive excitations of the reduced $AdS_5 \times S^5$ theory may be related to a trigonometric relativistic limit of the quantum-deformed $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ invariant R-matrix of [36, 37] which satisfies the YBE by construction.[6] The phase factor can be again fixed by unitarity and crossing and is the same as in the reduced $AdS_2 \times S^2$ and $AdS_3 \times S^3$ theories. The one-loop expansion of the resulting R-matrix indeed has a similar structure to the one-loop S-matrix that we found by direct computation.

One possibility is that the S-matrix for the perturbative excitations in a gauge-fixed Lagrangian is given by a certain non-unitary rotation of the quantum-deformed R-matrix. The violation of the YBE by the S-matrix computed directly from the Lagrangian may be related to some tension between gauge-fixing of the non-abelian H symmetry and the conservation of hidden charges. It is possible also that the physical excitations whose S-matrix is the quantum-deformed R-matrix may be some non-trivial gauge-invariant combinations of the Lagrangian fields.[7] At the moment, these are speculations and it is an open question as to what is the origin of the quantum deformation. An alternative approach based on semi-classical quantization of solitons [14, 15, 17, 48] may shed more light on this issue.

The structure of the rest of this paper is as follows. In section 2 we review the construction [1] of the Lagrangian for the perturbative excitations of the theory (1.1) near $g = \mathbf{1}$. We then present the result of the standard Feynman diagram computation of the one-loop S-matrix for the special cases corresponding to the reduced $AdS_2 \times S^2$, $AdS_3 \times S^3$ and $AdS_5 \times S^5$ theories.

In [3] it was shown that the reduced $AdS_2 \times S^2$ theory is equivalent to the $\mathcal{N} = 2$ supersymmetric sine-Gordon model. In section 3 we review the known construction of the exact S-matrix for the perturbative excitations of the $\mathcal{N} = 2$ supersymmetric sine-Gordon model and demonstrate its consistency with the direct one-loop computation. We also discuss various aspects of this S-matrix, such as the phase factor and the symmetries that will be useful for understanding the more complicated cases of the reduced $AdS_n \times S^n$ models with $n = 3, 5$.

In section 4 we consider the reduced $AdS_3 \times S^3$ theory. It is shown that as in the complex sine-Gordon theory [38, 28, 2] one can add a local counterterm to restore the satisfaction of the Yang-Baxter equation at the one-loop order. This counterterm also restores the group factorisation property of the S-matrix which then exhibits a quantum-deformed supersymmetry. Motivated by an analogy with the S-matrix of the reduced

[6] This is the limit [37] that has a similarity to the factorised tree-level S-matrix of [1]. For similar Lorentz invariant R-matrices see references in [36]. In particular, a relativistic quantum-deformed $\mathfrak{sl}(2|2)$ invariant R-matrix was constructed in [49]. One may expect [37] this R-matrix to be related to that discussed in section 6.2 of this paper.

[7] In the reduced $AdS_3 \times S^3$ and $AdS_5 \times S^5$ theories the quantum deformation parameter is, respectively, $q = e^{-2i\pi/k}$ and $e^{-i\pi/k}$, i.e. it is the coupling constant k that controls the deformation.

$AdS_2 \times S^2$ model and assuming quantum integrability as well as the quantum-deformed supersymmetry we then propose an expression for the exact quantum S-matrix for the perturbative excitations of this theory.

In section 5 the symmetries of the reduced $AdS_2 \times S^2$ and $AdS_3 \times S^3$ theories are reviewed and their origin in the Pohlmeyer reduction of the corresponding superstring theories is emphasized. By analogy, we then suggest that the physical symmetry of the reduced $AdS_5 \times S^5$ theory may be a quantum deformation of the $[\mathfrak{psu}(2|2)]^2 \ltimes \mathbb{R}^2$ superalgebra.

In section 6 we consider the S-matrix of the reduced $AdS_5 \times S^5$ theory. We demonstrate that the group-factorisation property is manifestly preserved at both the tree and one-loop level, indicating that no local counterterms are required here. At the same time, the S-matrix does not satisfy the YBE, and this cannot be repaired by adding local counterterms. Motivated by the discussion of symmetries in section 5 we investigate the similarity between the factorised one-loop S-matrix and the quantum-deformed $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^2$ symmetric R-matrix of [36] (which by construction satisfies the YBE). We extend the trigonometric relativistic classical limit of this R-matrix [37] to all orders in the $1/k$ expansion and conjecture that it should represent the quantum-deformed S-matrix of the reduced $AdS_5 \times S^5$ theory.

Some concluding remarks are made in section 7. Appendix A is an extension of the review of the $\mathcal{N} = 2$ supersymmetric sine-Gordon S-matrix in section 3. In appendix B we investigate the symmetries of the reduced $AdS_3 \times S^3$ theory. In appendix C we discuss the origin of the one-loop counterterm required in section 4 and demonstrate how it can be derived from a functional determinant. Appendices D and E give a detailed form of the factorised S-matrices of the reduced $AdS_3 \times S^3$ and $AdS_5 \times S^5$ theories. In appendix F we extend the classical relativistic trigonometric limit [37] of the quantum-deformed $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^2$ R-matrix of [36] to all orders. Finally, in appendix G we present an updated discussion of the S-matrix of the bosonic $G/H = SO(N+1)/SO(N)$ generalized sine-Gordon theories studied earlier in [2].

2 Perturbative S-matrix to one-loop order

In this section we find the one-loop S-matrix for the perturbative excitations of the Pohlmeyer reduction of the superstring theory on $AdS_n \times S^n$ for $n = 2, 3, 5$ (the tree-level term in this S-matrix was found in [1]). The reduced theories are fermionic extensions of a generalised sine-Gordon model (gauged WZW theory with an integrable potential) [3, 5] whose Lagrangian is given in (1.1).

2.1 General setup

We parametrise the group field in (1.1) as

$$g = \exp(X + \xi)\,, \qquad \text{where} \quad X \in \mathfrak{g} \ominus \mathfrak{h}\,, \quad \xi \in \mathfrak{h}\,. \tag{2.1}$$

Following [1], we fix the $A_+ = 0$ gauge in (1.1) and integrate over A_-. The resulting constraint equation allows us to perturbatively solve for ξ (2.1), leaving the required $2(n-1) + 2(n-1)$ massive degrees of freedom.

With the help of integration by parts and field redefinitions that amount to use of the linearised equations of motion we get the following local quartic Lagrangian [1]

$$\begin{aligned} L = \frac{k}{4\pi}\ \mathrm{STr}\Big(& \frac{1}{2}\partial_+ X \partial_- X - \frac{\mu^2}{2}X^2 + \Psi_{_L} T \partial_+ \Psi_{_L} + \Psi_{_R} T \partial_- \Psi_{_R} + \mu \Psi_{_L}\Psi_{_R} \\ & + \frac{1}{12}[X,\, \partial_+ X][X,\, \partial_- X] + \frac{\mu^2}{24}[X,\, [X,\, T]]^2 \\ & - \frac{1}{4}[\Psi_{_L} T,\, \Psi_{_L}][X,\, \partial_+ X] - \frac{1}{4}[\Psi_{_R},\, T\Psi_{_R}][X,\, \partial_- X] - \frac{\mu}{2}[X,\, \Psi_{_R}][X,\, \Psi_{_L}] \\ & + \frac{1}{2}[\Psi_{_L} T,\, \Psi_{_L}][\Psi_{_R},\, T\Psi_{_R}] + \dots \Big)\,. \end{aligned} \tag{2.2}$$

This quartic Lagrangian will be sufficient to compute the one-loop 2-particle S-matrix (see below). Considering the Pohlmeyer reduction of superstring theory on $AdS_5 \times S^5$, where the supergroup $\hat{F} = PSU(2,2|4)$ and $G = Sp(2,2) \times Sp(4)$, one can expand (2.2) in components [1] (rescaling the fields by $\sqrt{\frac{4\pi}{k}}$, $L \to \mathcal{L}_5$)

$$\mathcal{L}_5 \;=\; \frac{1}{2}\partial_+ Y_m \partial_- Y_m - \frac{\mu^2}{2} Y_m Y_m + \frac{1}{2}\partial_+ Z_m \partial_- Z_m - \frac{\mu^2}{2} Z_m Z_m$$

$$
\begin{aligned}
&+\frac{i}{2}\zeta_{Lm}\partial_+\zeta_{Lm}+\frac{i}{2}\zeta_{Rm}\partial_-\zeta_{Rm}+i\mu\zeta_{Rm}\zeta_{Lm}+\frac{i}{2}\chi_{Lm}\partial_+\chi_{Lm}+\frac{i}{2}\chi_{Rm}\partial_-\chi_{Rm}+i\mu\chi_{Rm}\chi_{Lm}\\
&+\frac{\pi}{2k}\Big[-\frac{2}{3}Y_mY_m\partial_+Y_n\partial_-Y_n+\frac{2}{3}Y_m\partial_+Y_mY_n\partial_-Y_n-\frac{\mu^2}{3}Y_mY_mY_nY_n\\
&\qquad+\frac{2}{3}Z_mZ_m\partial_+Z_n\partial_-Z_n-\frac{2}{3}Z_m\partial_+Z_mZ_n\partial_-Z_n+\frac{\mu^2}{3}Z_mZ_mZ_nZ_n\\
&\qquad+\frac{i}{2}(\gamma_{mnpq}+\epsilon_{mnpq})(\zeta_{Lm}\zeta_{Ln}Y_p\partial_+Y_q+\zeta_{Rm}\zeta_{Rn}Y_p\partial_-Y_q\\
&\qquad\qquad\qquad\qquad-\chi_{Lm}\chi_{Ln}Z_p\partial_+Z_q-\chi_{Rm}\chi_{Rn}Z_p\partial_-Z_q)\\
&\qquad+\frac{i}{2}(\gamma_{mnpq}-\epsilon_{mnpq})(\chi_{Lm}\chi_{Ln}Y_p\partial_+Y_q+\chi_{Rm}\chi_{Rn}Y_p\partial_-Y_q\\
&\qquad\qquad\qquad\qquad-\zeta_{Lm}\zeta_{Ln}Z_p\partial_+Z_q-\zeta_{Rm}\zeta_{Rn}Z_p\partial_-Z_q)\\
&\qquad+i\mu(\zeta_{Rm}\zeta_{Lm}+\chi_{Rm}\chi_{Lm})(Y_nY_n-Z_nZ_n)\\
&\qquad-2i\mu(\epsilon_{mnpq}+\delta_{mn}\delta_{pq}-\delta_{mp}\delta_{nq}-\delta_{mq}\delta_{np})(\zeta_{Rm}\chi_{Ln}Y_pZ_q-\chi_{Rm}\zeta_{Ln}Z_pY_q)\\
&\qquad+\epsilon_{mnpq}(\zeta_{Rm}\zeta_{Rn}\zeta_{Lp}\zeta_{Lq}-\chi_{Rm}\chi_{Rn}\chi_{Lp}\chi_{Lq})\Big]+\mathcal{O}(k^{-2})\,.
\end{aligned}
\tag{2.3}
$$

m,n,p,q are $SO(4)$ vector indices.[8] We have also defined the $SO(N)$ tensor γ_{mnpq} as

$$
\gamma_{mnpq}=\delta_{mp}\delta_{nq}-\delta_{mq}\delta_{np}\,. \tag{2.4}
$$

The fields Y_m and Z_m (which are components of X in (2.2)) are bosonic, while the fields ζ_{Rm}, ζ_{Lm}, χ_{Rm} and χ_{Lm} (originating from Ψ_L, Ψ_R) are 2-d Majorana-Weyl fermions. This Lagrangian describes $8+8$ massive degrees of freedom.

Up to a scaling ambiguity in k, the analogous Lagrangians for the reduced $AdS_2\times S^2$ and $AdS_3\times S^3$ theories are given by restricting the indices to m,n,p,q to take the values 1 and $1,2$ respectively.[9] The expansion of these Lagrangians to quartic order agrees with those that arise from the $A_+=0$ gauge treatment up to field redefinitions [1].

Below we compute the one-loop two-particle S-matrix arising from the Lagrangian (2.3) and its $AdS_2\times S^2$ and $AdS_3\times S^3$ truncations. Following [1] we use Feynman diagrams and standard perturbative quantum field theory. From the quadratic part of the action one can construct the asymptotic states for which the spatial momentum and energy eigenvalues are related by the usual relativistic dispersion relation

$$
E=\sqrt{p^2+\mu^2}\,. \tag{2.5}
$$

In 2-d relativistic theories it is convenient to consider the corresponding rapidity ϑ, related to the on-shell spatial momentum as

$$
p=\mu\sinh\vartheta\,. \tag{2.6}
$$

We will label the on-shell momenta of the incoming states as p_1 and p_2, with the corresponding rapidities ϑ_1 and ϑ_2. As we are considering integrable theories, the outgoing states should have the same momenta as the incoming states and as the theories are relativistic the S-matrix should only depend on the difference of the rapidities

$$
\theta=\vartheta_1-\vartheta_2\,. \tag{2.7}
$$

[8] Note that for notational convenience we are using $SO(N)$ indices, see [1] for conventions. The group G here is $Sp(2,2)\times Sp(4)$ and thus the index of the fundamental representation ν in (1.1) is equal to $\frac{1}{2}$. For a discussion of the normalisation of k see footnote 9 below.

[9] For the reduced $AdS_3\times S^3$ theory we should also rescale $k\to\frac{k}{2}$. The reason being that in this case $G=SU(1,1)\times SU(2)$ and the dual Coxeter number of $SU(2)$, $c_{SU(2)}=2$ is twice that of $SO(3)$, $c_{SO(3)}=1$. For the reduced $AdS_5\times S^5$ theory we have $G=Sp(2,2)\times Sp(4)$. The dual Coxeter number of $Sp(4)$, $c_{Sp(4)}=3$ is equal to that of $SO(5)$, $c_{SO(5)}=3$. Therefore the reduced $AdS_3\times S^3$ theory should have $k\to\frac{k}{2}$ compared to the $G/H=SO(3)/SO(2)$ theory in [2], whereas the reduced $AdS_5\times S^5$ model should have the same normalisation of k as the $G/H=SO(5)/SO(4)$ theory of [2]. For the reduced $AdS_2\times S^2$ case the group G is abelian and thus there is no quantization of k, i.e. it can be arbitrarily rescaled. For convenience we will assume the same normalisation as in the reduced $AdS_3\times S^3$ theory, i.e. we will rescale $k\to\frac{k}{2}$ in (2.3). This is also the same normalisation as when one takes $\nu=\frac{1}{2}$ in (1.1).

In the reduced $AdS_n \times S^n$ theories there are $16\,(\times(n-1)^2)$ two-particle states given by

$$\begin{array}{llll}
|Y_m(p_1)Y_n(p_2)\rangle\,, & |Z_m(p_1)Z_n(p_2)\rangle\,, & |\zeta_m(p_1)\zeta_n(p_2)\rangle\,, & |\chi_m(p_1)\chi_n(p_2)\rangle\,,\\
|Y_m(p_1)\zeta_n(p_2)\rangle\,, & |\zeta_m(p_1)Y_n(p_2)\rangle\,, & |Z_m(p_1)\chi_n(p_2)\rangle\,, & |\chi_m(p_1)Z_n(p_2)\rangle\,,\\
|Y_m(p_1)\chi_n(p_2)\rangle\,, & |\chi_m(p_1)Y_n(p_2)\rangle\,, & |Z_m(p_1)\zeta_n(p_2)\rangle\,, & |\zeta_m(p_1)Z_n(p_2)\rangle\,,\\
|Y_m(p_1)Z_n(p_2)\rangle\,, & |Z_m(p_1)Y_n(p_2)\rangle\,, & |\zeta_m(p_1)\chi_n(p_2)\rangle\,, & |\chi_m(p_1)\zeta_n(p_2)\rangle\,.
\end{array} \tag{2.8}$$

Naively we have 256 amplitudes in the two-particle S-matrix; however, from the Lagrangian (2.3) we see that at the tree level there are selection rules such that the four rows of two-particle states in (2.8) only scatter amongst themselves. This is a consequence of symmetries that are not manifest in the Lagrangian (2.3), as such these selection rules apply beyond the tree level. We therefore have $4^2 \times 4 = 64$ non-zero scattering processes. The symmetries are discussed in greater detail for each of the particular theories in later sections.

We will list 40 of these amplitudes for each of the $AdS_n \times S^n$ reduced theories, from which the remaining 24 ones can be easily derived. For example, to compute $B_{mnpq}(\theta)$ in

$$\mathbb{S}\,|\zeta_m(p_1)Y_n(p_2)\rangle = B_{mnpq}(\theta)\,|Y_p(p_1)\zeta_q(p_2)\rangle + \dots\,, \tag{2.9}$$

where $\mathbb{S}$ is the 2-particle S-matrix operator and the dots stand for other possible terms, we may use the fact that we know $A_{mnpq}(\theta)$ in

$$\begin{aligned}
\mathbb{S}\,|Y_m(p_1)\zeta_n(p_2)\rangle = A_{mnpq}(\theta)\,|\zeta_p(p_1)Y_q(p_2)\rangle + \dots \;\Rightarrow\; & \mathbb{S}\,|\zeta_m(p_2)Y_n(p_1)\rangle = A_{nmqp}(\theta)\,|Y_p(p_2)\zeta_q(p_1)\rangle + \dots \\
\Rightarrow\; & \mathbb{S}\,|\zeta_m(p_1)Y_n(p_2)\rangle = A^*_{nmqp}(-\theta)\,|Y_p(p_1)\zeta_q(p_2)\rangle + \dots\,,
\end{aligned} \tag{2.10}$$

implying that

$$B_{mnpq}(\theta) = A^*_{nmqp}(-\theta)\,. \tag{2.11}$$

Clearly, if two fermions are passing through each other in (2.10) we pick up a factor of -1.

As the reduced $AdS_n \times S^n$ theories are classically integrable (there exists a Lax connection [3]) we expect them to be quantum-integrable so that the two-particle S-matrix operator should satisfy the Yang-Baxter equation

$$\mathbb{S}_{12}(\theta_{12})\mathbb{S}_{13}(\theta_{13})\mathbb{S}_{23}(\theta_{23}) = \mathbb{S}_{23}(\theta_{23})\mathbb{S}_{13}(\theta_{13})\mathbb{S}_{12}(\theta_{12}) \tag{2.12}$$

Here the triple operator products should be understood as acting on a three particle state with rapidities ϑ_1, ϑ_2, ϑ_3. The subscripts on $\mathbb{S}$ label which particles in the state it is acting on, while the quantities θ_{ij} denote the rapidity differences,

$$\theta_{ij} = \vartheta_i - \vartheta_j\,. \tag{2.13}$$

We therefore have the usual relativistic relation

$$\theta_{13} = \theta_{12} + \theta_{23}\,. \tag{2.14}$$

For the reduced $AdS_n \times S^n$ theories where we are scattering fermions and bosons (2.12) will have some fermionic grading as well.

If we consider the S-matrix to a particular order in the weak coupling ($\frac{1}{k} \ll 1$) expansion, say $\mathcal{O}(k^{-n-1})$, then the corresponding order of the Yang-Baxter equation is $\mathcal{O}(k^{-n})$: the contribution of the $\mathcal{O}(k^{-n})$ part of the S-matrix to the $\mathcal{O}(k^{-n})$ part of the Yang-Baxter equation (2.12) vanishes trivially.

2.2 One-loop results

Below we will present the results of the direct one-loop computation of the two-particle S-matrix for the reduced $AdS_n \times S^n$ ($n = 2, 3, 5$) theories. We use standard Feynman diagram techniques starting with the Lagrangian (2.3) or its various truncations.

The reduced $AdS_5 \times S^5$ theory is UV-finite [20]. Also, it was shown in [3] that the reduced $AdS_2 \times S^2$ theory is equivalent to the $\mathcal{N} = 2$ supersymmetric sine-Gordon theory and thus is UV-finite as well. Semiclassical computations as in [21] provide a check that the reduced $AdS_3 \times S^3$ theory is also UV-finite, at least to the two-loop order. Thus in contrast to the purely bosonic theories there is no renormalisation of the mass parameter μ.

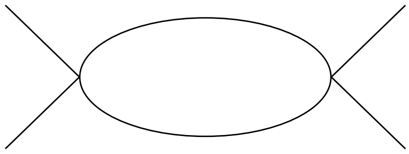

Figure 1: Bubble diagram

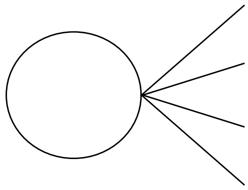

Figure 2: Tadpole diagram

To compute the one-loop S-matrix one need only consider the Feynman diagrams of the form in Fig.1. This is because the finite part of the tadpole diagrams in Fig.2 arising from the sextic terms in the Lagrangian (2.3) will vanish in 2-d. Due to the form of the fermion-boson and fermion-fermion interactions the (gauged WZW model based) theories we consider here are naturally defined only in 2 dimensions so that dimensional regularisation is not suitable. Instead we assume a direct momentum cut-off and ignore divergent terms. They should cancel against the contributions of tadpole diagrams coming from the sextic tadpoles, as the theories are UV-finite.

Below we will list the expressions for the one-loop S-matrices in each of the three reduced $AdS_n \times S^n$ theories. It will be useful to extract the factor

$$p_0(\theta, k\,; c) = 1 + \ c\ \frac{\pi \operatorname{cosech}\theta}{2k^2}\Big(i\big[2 + (i\pi - 2\theta)\coth\theta\big] - \pi \operatorname{cosech}\theta\Big) \tag{2.15}$$

from the one-loop S-matrix (with different values of the constant c depending on the theory). This will set the $Y_m Z_n \to Y_m Z_n$ and $\zeta_m \chi_n \to \zeta_m \chi_n$ amplitudes equal to one, at least to one-loop order.

2.2.1 Reduced $AdS_2 \times S^2$ theory

Let us start with the Lagrangian (2.3) and restrict the indices m, n, p, q to only take a single value and rescale $k \to \frac{k}{2}$. The resulting Lagrangian is then that of the reduced $AdS_2 \times S^2$ theory [3, 1] expanded to quartic order

$$\begin{aligned}
\mathcal{L}_2 \ = \ & \frac{1}{2}\partial_+ Y \partial_- Y - \frac{\mu^2}{2}Y^2 + \frac{1}{2}\partial_+ Z \partial_- Z - \frac{\mu^2}{2}Z^2 \\
& + \frac{i}{2}\zeta_L \partial_+ \zeta_L + \frac{i}{2}\zeta_R \partial_- \zeta_R + i\mu\zeta_R\zeta_L + \frac{i}{2}\chi_L \partial_+ \chi_L + \frac{i}{2}\chi_R \partial_- \chi_R + i\mu\chi_R\chi_L \\
& + \frac{\pi}{k}\Big[-\frac{\mu^2}{3}Y^4 + \frac{\mu^2}{3}Z^4 + i\mu(\zeta_R\zeta_L + \chi_R\chi_L)(Y^2 - Z^2) + 2i\mu(\zeta_R\chi_L - \chi_R\zeta_L)YZ\Big] + \mathcal{O}(k^{-2})\,.
\end{aligned} \tag{2.16}$$

The resulting one-loop S-matrix is found to have the following structure:

Boson-Boson

$$\begin{aligned}
\mathbb{S}\,|Y(p_1)Y(p_2)\rangle = & f_1(\theta,k)\,|Y(p_1)Y(p_2)\rangle + f_5(\theta,k)\,|Z(p_1)Z(p_2)\rangle \\
& + f_6(\theta,k)\,|\zeta(p_1)\zeta(p_2)\rangle + f_6(\theta,k)\,|\chi(p_1)\chi(p_2)\rangle
\end{aligned}$$

$$\begin{aligned}\mathbb{S}\,|Z(p_1)Z(p_2)\rangle =& f_1(\theta,-k)\,|Z(p_1)Z(p_2)\rangle + f_5(\theta,-k)\,|Y(p_1)Y(p_2)\rangle\\ &+ f_6(\theta,-k)\,|\chi(p_1)\chi(p_2)\rangle + f_6(\theta,-k)\,|\zeta(p_1)\zeta(p_2)\rangle\\ \mathbb{S}\,|Y(p_1)Z(p_2)\rangle =& f_3(\theta,k)\,|Y(p_1)Z(p_2)\rangle + f_5(i\pi-\theta,k)\,|Z(p_1)Y(p_2)\rangle\\ &+ f_7(\theta,k)\,|\zeta(p_1)\chi(p_2)\rangle - f_7(\theta,k)\,|\chi(p_1)\zeta(p_2)\rangle\end{aligned}$$

Boson-Fermion

$$\begin{aligned}\mathbb{S}\,|Y(p_1)\zeta(p_2)\rangle =& f_4(\theta,k)\,|Y(p_1)\zeta(p_2)\rangle + if_6(i\pi-\theta,k)\,|\zeta(p_1)Y(p_2)\rangle\\ &+ f_8(\theta,k)\,|Z(p_1)\chi(p_2)\rangle + if_7(i\pi-\theta,k)\,|\chi(p_1)Z(p_2)\rangle\\ \mathbb{S}\,|Y(p_1)\chi(p_2)\rangle =& f_4(\theta,k)\,|Y(p_1)\chi(p_2)\rangle + if_6(i\pi-\theta,k)\,|\chi(p_1)Y(p_2)\rangle\\ &- f_8(\theta,k)\,|Z(p_1)\zeta(p_2)\rangle - if_7(i\pi-\theta,k)\,|\zeta(p_1)Z(p_2)\rangle\\ \mathbb{S}\,|Z(p_1)\zeta(p_2)\rangle =& f_4(\theta,-k)\,|Z(p_1)\zeta(p_2)\rangle + if_6(i\pi-\theta,-k)\,|\zeta(p_1)Z(p_2)\rangle\\ &- f_8(\theta,-k)\,|Y(p_1)\chi(p_2)\rangle - if_7(i\pi-\theta,-k)\,|\chi(p_1)Y(p_2)\rangle\\ \mathbb{S}\,|Z(p_1)\chi(p_2)\rangle =& f_4(\theta,-k)\,|Z(p_1)\chi(p_2)\rangle + if_6(i\pi-\theta,-k)\,|\chi(p_1)Z(p_2)\rangle\\ &+ f_8(\theta,-k)\,|Y(p_1)\zeta(p_2)\rangle + if_7(i\pi-\theta,-k)\,|\zeta(p_1)Y(p_2)\rangle\end{aligned}$$

Fermion-Fermion

$$\begin{aligned}\mathbb{S}\,|\zeta(p_1)\zeta(p_2)\rangle =&\ f_2(\theta,k)\,|\zeta(p_1)\zeta(p_2)\rangle - f_5(\theta,k)\,|\chi(p_1)\chi(p_2)\rangle\\ &+ f_6(\theta,k)\,|Y(p_1)Y(p_2)\rangle + f_6(\theta,-k)\,|Z(p_1)Z(p_2)\rangle\\ \mathbb{S}\,|\chi(p_1)\chi(p_2)\rangle =& f_2(\theta,-k)\,|\chi(p_1)\chi(p_2)\rangle - f_5(\theta,-k)\,|\zeta(p_1)\zeta(p_2)\rangle\\ &+ f_6(\theta,-k)\,|Z(p_1)Z(p_2)\rangle + f_6(\theta,k)\,|Y(p_1)Y(p_2)\rangle\\ \mathbb{S}\,|\zeta(p_1)\chi(p_2)\rangle =& f_3(\theta,k)\,|\zeta(p_1)\chi(p_2)\rangle - f_5(i\pi-\theta,k)\,|\chi(p_1)\zeta(p_2)\rangle\\ &+ f_7(\theta,k)\,|Y(p_1)Z(p_2)\rangle + f_7(\theta,k)\,|Z(p_1)Y(p_2)\rangle\end{aligned}$$

Functions

$$\hat{f}_1(\theta,k) =1 - \frac{2i\pi}{k}\,\text{cosech}\,\theta - \frac{\pi\,\text{cosech}\,\theta}{k^2} + \mathcal{O}(k^{-3})\,, \qquad \hat{f}_2(\theta,k) =1 + \frac{\pi\,\text{cosech}^2\theta}{k^2} + \mathcal{O}(k^{-3})\,,$$

$$\hat{f}_3(\theta,k) =1 + \mathcal{O}(k^{-3})\,, \qquad \hat{f}_4(\theta,k) =1 - \frac{i\pi}{k}\,\text{cosech}\,\theta + \mathcal{O}(k^{-3})\,,$$

$$\hat{f}_5(\theta,k) =\frac{\pi^2}{4k^2}\,\text{sech}^2\frac{\theta}{2} + \mathcal{O}(k^{-3})\,, \qquad \hat{f}_6(\theta,k) =\frac{i\pi}{2k}\,\text{sech}\,\frac{\theta}{2} + \frac{\pi^2}{2k^2}\,\text{cosech}\,\theta\,\text{sech}\,\frac{\theta}{2} + \mathcal{O}(k^{-3})\,,$$

$$\hat{f}_7(\theta,k) =\frac{i\pi}{2k}\,\text{cosech}\,\frac{\theta}{2} + \mathcal{O}(k^{-3})\,, \qquad \hat{f}_8(\theta,k) =-\frac{\pi^2}{2k^2}\,\text{cosech}\,\theta + \mathcal{O}(k^{-3})\,.$$

The 8 functions $\hat{f}_i$ are related to f_i entering $\mathbb{S}$ by the phase factor $p_0(\theta,k\,;1)$ defined in (2.15)

$$f_i(\theta,k) = p_0(\theta,k\,;1)\,\hat{f}_i(\theta,k)\ . \tag{2.17}$$

Here the $1/k$ terms in f_i represent tree-level contributions to the 2-particle S-matrix (k is the overall coefficient in (1.1), see also (2.3)) and the $1/k^2$ terms – the one-loop contributions.

2.2.2 Reduced $AdS_3 \times S^3$ theory

Here we start again with the Lagrangian (2.3), rescaling $k \to \frac{k}{2}$ and restricting the indices m, n, p, q to take the values 1 and 2 (so they become $SO(2)$ vector indices). The resulting Lagrangian is then that of the reduced $AdS_3 \times S^3$ theory [5, 1] expanded to quartic order

$$\begin{aligned}\mathcal{L}_3 \;=\;& \frac{1}{2}\partial_+Y_m\partial_-Y_m - \frac{\mu^2}{2}Y_mY_m + \frac{1}{2}\partial_+Z_m\partial_-Z_m - \frac{\mu^2}{2}Z_mZ_m\\ &+\frac{i}{2}\zeta_{_{L}m}\partial_+\zeta_{_{L}m} + \frac{i}{2}\zeta_{_{R}m}\partial_-\zeta_{_{R}m} + i\mu\zeta_{_{R}m}\zeta_{_{L}m} + \frac{i}{2}\chi_{_{L}m}\partial_+\chi_{_{L}m} + \frac{i}{2}\chi_{_{R}m}\partial_-\chi_{_{R}m} + i\mu\chi_{_{R}m}\chi_{_{L}m}\\ &+\frac{\pi}{k}\Big[-\frac{2}{3}Y_mY_m\partial_+Y_n\partial_-Y_n + \frac{2}{3}Y_m\partial_+Y_mY_n\partial_-Y_n - \frac{\mu^2}{3}Y_mY_mY_nY_n\end{aligned}$$

$$
\begin{aligned}
&+\frac{2}{3}Z_mZ_m\partial_+Z_n\partial_-Z_n-\frac{2}{3}Z_m\partial_+Z_mZ_n\partial_-Z_n+\frac{\mu^2}{3}Z_mZ_mZ_nZ_n\\
&+\frac{i}{2}\gamma_{mnpq}(\zeta_{Lm}\zeta_{Ln}Y_p\partial_+Y_q+\zeta_{Rm}\zeta_{Rn}Y_p\partial_-Y_q\\
&\qquad\qquad -\chi_{Lm}\chi_{Ln}Z_p\partial_+Z_q-\chi_{Rm}\chi_{Rn}Z_p\partial_-Z_q)\\
&+\frac{i}{2}\gamma_{mnpq}(\chi_{Lm}\chi_{Ln}Y_p\partial_+Y_q+\chi_{Rm}\chi_{Rn}Y_p\partial_-Y_q\\
&\qquad\qquad -\zeta_{Lm}\zeta_{Ln}Z_p\partial_+Z_q-\zeta_{Rm}\zeta_{Rn}Z_p\partial_-Z_q)\\
&+i\mu(\zeta_{Rm}\zeta_{Lm}+\chi_{Rm}\chi_{Lm})(Y_nY_n-Z_nZ_n)\\
&-2i\mu(\delta_{mn}\delta_{pq}-\delta_{mp}\delta_{nq}-\delta_{mq}\delta_{np})(\zeta_{Rm}\chi_{Ln}Y_pZ_q-\chi_{Rm}\zeta_{Ln}Z_pY_q)\Big]+\mathcal{O}(k^{-2})
\end{aligned}
\tag{2.18}
$$

The corresponding one-loop S-matrix is found to be:

Boson-Boson

$$
\begin{aligned}
\mathbb{S}\,|Y_m(p_1)Y_n(p_2)\rangle =&f_{1mnpq}(\theta,k)\,|Y_p(p_1)Y_q(p_2)\rangle+f_{5mnpq}(\theta,k)\,|Z_p(p_1)Z_q(p_2)\rangle\\
&+f_{6mnpq}(\theta,k)\,|\zeta_p(p_1)\zeta_q(p_2)\rangle+f_{6mnpq}(\theta,k)\,|\chi_p(p_1)\chi_q(p_2)\rangle\\
\mathbb{S}\,|Z_m(p_1)Z_n(p_2)\rangle =&f_{1mnpq}(\theta,-k)\,|Z_p(p_1)Z_q(p_2)\rangle+f_{5mnpq}(\theta,-k)\,|Y_p(p_1)Y_q(p_2)\rangle\\
&+f_{6mnpq}(\theta,-k)\,|\chi_p(p_1)\chi_q(p_2)\rangle+f_{6mnpq}(\theta,-k)\,|\zeta_p(p_1)\zeta_q(p_2)\rangle\\
\mathbb{S}\,|Y_m(p_1)Z_n(p_2)\rangle =&f_{3mnpq}(\theta,k)\,|Y_p(p_1)Z_q(p_2)\rangle+f_{5mqpn}(i\pi-\theta,k)\,|Z_p(p_1)Y_q(p_2)\rangle\\
&+f_{7mnpq}(\theta,k)\,|\zeta_p(p_1)\chi_q(p_2)\rangle-f_{7mnpq}(\theta,k)\,|\chi_p(p_1)\zeta_q(p_2)\rangle
\end{aligned}
$$

Boson-Fermion

$$
\begin{aligned}
\mathbb{S}\,|Y_m(p_1)\zeta_n(p_2)\rangle =&f_{4mnpq}(\theta,k)\,|Y_p(p_1)\zeta_q(p_2)\rangle+if_{6mqpn}(i\pi-\theta,k)\,|\zeta_p(p_1)Y_q(p_2)\rangle\\
&+f_{8mnpq}(\theta,k)\,|Z_p(p_1)\chi_q(p_2)\rangle+if_{7mqpn}(i\pi-\theta,k)\,|\chi_p(p_1)Z_q(p_2)\rangle\\
\mathbb{S}\,|Y_m(p_1)\chi_n(p_2)\rangle =&f_{4mnpq}(\theta,k)\,|Y_p(p_1)\chi_q(p_2)\rangle+if_{6mqpn}(i\pi-\theta,k)\,|\chi_p(p_1)Y_q(p_2)\rangle\\
&-f_{8mnpq}(\theta,k)\,|Z_p(p_1)\zeta_q(p_2)\rangle-if_{7mqpn}(i\pi-\theta,k)\,|\zeta_p(p_1)Z_q(p_2)\rangle\\
\mathbb{S}\,|Z_m(p_1)\zeta_n(p_2)\rangle =&f_{4mnpq}(\theta,-k)\,|Z_p(p_1)\zeta_q(p_2)\rangle+if_{6mqpn}(i\pi-\theta,-k)\,|\zeta_p(p_1)Z_q(p_2)\rangle\\
&-f_{8mnpq}(\theta,-k)\,|Y_p(p_1)\chi_q(p_2)\rangle-if_{7mqpn}(i\pi-\theta,-k)\,|\chi_p(p_1)Y_q(p_2)\rangle\\
\mathbb{S}\,|Z_m(p_1)\chi_n(p_2)\rangle =&f_{4mnpq}(\theta,-k)\,|Z_p(p_1)\chi_q(p_2)\rangle+if_{6mqpn}(i\pi-\theta,-k)\,|\chi_p(p_1)Z_q(p_2)\rangle\\
&+f_{8mnpq}(\theta,-k)\,|Y_p(p_1)\zeta_q(p_2)\rangle+if_{7mqpn}(i\pi-\theta,-k)\,|\zeta_p(p_1)Y_q(p_2)\rangle
\end{aligned}
$$

Fermion-Fermion

$$
\begin{aligned}
\mathbb{S}\,|\zeta_m(p_1)\zeta_n(p_2)\rangle =&f_{2mnpq}(\theta,k)\,|\zeta_p(p_1)\zeta_q(p_2)\rangle-f_{\tilde{5}mnpq}(\theta,k)\,|\chi_p(p_1)\chi_q(p_2)\rangle\\
&+f_{6mnpq}(\theta,k)\,|Y_p(p_1)Y_q(p_2)\rangle+f_{6mnpq}(\theta,-k)\,|Z_p(p_1)Z_q(p_2)\rangle\\
\mathbb{S}\,|\chi_m(p_1)\chi_n(p_2)\rangle =&f_{2mnpq}(\theta,-k)\,|\chi_p(p_1)\chi_q(p_2)\rangle-f_{\tilde{5}mnpq}(\theta,-k)\,|\zeta_p(p_1)\zeta_q(p_2)\rangle\\
&+f_{6mnpq}(\theta,-k)\,|Z_p(p_1)Z_q(p_2)\rangle+f_{6mnpq}(\theta,k)\,|Y_p(p_1)Y_q(p_2)\rangle\\
\mathbb{S}\,|\zeta_m(p_1)\chi_n(p_2)\rangle =&f_{\tilde{3}mnpq}(\theta,k)\,|\zeta_p(p_1)\chi_q(p_2)\rangle-f_{\tilde{5}mqpn}(i\pi-\theta,k)\,|\chi_p(p_1)\zeta_q(p_2)\rangle\\
&+f_{7mnpq}(\theta,k)\,|Y_p(p_1)Z_q(p_2)\rangle+f_{7mnpq}(\theta,k)\,|Z_p(p_1)Y_q(p_2)\rangle
\end{aligned}
$$

Functions

$$
\begin{aligned}
\hat{f}_{1mnpq}(\theta,k) =&\delta_{mp}\delta_{nq}\Big(1-\frac{2i\pi}{k}\operatorname{cosech}\theta-\frac{2\pi^2}{k^2}\coth^2\theta\Big)\\
&+\epsilon_{mp}\epsilon_{nq}\Big(\frac{2i\pi}{k}\coth\theta-\frac{i\pi}{k^2}(i\pi-2\theta)\operatorname{cosech}^2\theta+\frac{3\pi^2}{k^2}\coth\theta\operatorname{cosech}\theta\Big)\\
&\underline{-\frac{i\pi}{k^2}\Big(\epsilon_{mn}\epsilon_{pq}(\operatorname{cosech}\theta-\coth\theta)+\epsilon_{mq}\epsilon_{pn}(\operatorname{cosech}\theta+\coth\theta)\Big)}+\mathcal{O}(k^{-3}\ ,)\\
\hat{f}_{2mnpq}(\theta,k) =&\delta_{mp}\delta_{nq}\Big(1+\frac{2\pi^2}{k^2}\operatorname{cosech}^2\theta\Big)\\
&+\epsilon_{mp}\epsilon_{nq}\Big(-\frac{i\pi}{k^2}(i\pi-2\theta)\operatorname{cosech}^2\theta-\frac{\pi^2}{k^2}\coth\theta\operatorname{cosech}\theta\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad \underline{+\frac{2i\pi}{k^2}\,\mathrm{cosech}\,\theta\Big(\epsilon_{mn}\epsilon_{pq}+\epsilon_{mq}\epsilon_{pn}\Big)}+\mathcal{O}(k^{-3})\ ,\\
\hat f_{3mnpq}(\theta,k)=&\hat f_{\bar 3mnpq}(\theta,k)=\delta_{mp}\delta_{nq}+\epsilon_{mp}\epsilon_{nq}\Big(-\frac{i\pi}{k^2}(i\pi-2\theta)\,\mathrm{cosech}^2\theta+\frac{\pi^2}{k^2}\coth\theta\,\mathrm{cosech}\,\theta\Big)+\mathcal{O}(k^{-3})\ ,\\
\hat f_{4mnpq}(\theta,k)=&\delta_{mp}\delta_{nq}\Big(1-\frac{i\pi}{k}\,\mathrm{cosech}\,\theta-\frac{\pi^2}{2k^2}\Big)\\
&+\epsilon_{mp}\epsilon_{nq}\Big(\frac{i\pi}{k}\coth\theta-\frac{i\pi}{k^2}(i\pi-2\theta)\,\mathrm{cosech}^2\theta+\frac{\pi^2}{k^2}\coth\theta\,\mathrm{cosech}\,\theta\Big)+\mathcal{O}(k^{-3})\ ,\\
\hat f_{5mnpq}(\theta,k)=&\frac{\pi^2}{2k^2}\,\mathrm{sech}^2\frac{\theta}{2}(\delta_{mn}\delta_{pq}+\epsilon_{mn}\epsilon_{pq})\underline{-\frac{i\pi}{k^2}\epsilon_{mn}\epsilon_{pq}(\,\mathrm{cosech}\,\theta-\coth\theta)}+\mathcal{O}(k^{-3})\ ,\\
\hat f_{\bar 5mnpq}(\theta,k)=&\frac{\pi^2}{2k^2}\,\mathrm{sech}^2\frac{\theta}{2}(\delta_{mn}\delta_{pq}+\epsilon_{mn}\epsilon_{pq})\underline{-\frac{2i\pi}{k^2}\epsilon_{mn}\epsilon_{pq}\,\mathrm{cosech}\,\theta}+\mathcal{O}(k^{-3})\ ,\\
\hat f_{6mnpq}(\theta,k)=&\Big(\frac{i\pi}{2k}\,\mathrm{sech}\,\frac{\theta}{2}-\frac{\pi^2}{2k^2}\,\mathrm{sech}\,\frac{\theta}{2}\tanh\frac{\theta}{2}\Big)(\delta_{mn}\delta_{pq}+\epsilon_{mn}\epsilon_{pq})\underline{+\frac{i\pi}{k^2}\epsilon_{mn}\epsilon_{pq}\,\mathrm{sech}\,\frac{\theta}{2}}+\mathcal{O}(k^{-3})\ ,\\
\hat f_{7mnpq}(\theta,k)=&\frac{i\pi}{2k}\,\mathrm{cosech}\,\frac{\theta}{2}\big(-\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}+\delta_{mq}\delta_{np}\big)+\mathcal{O}(k^{-3})\ ,\\
\hat f_{8mnpq}(\theta,k)=&\mathcal{O}(k^{-3})\ .
\end{aligned}
$$

As in the previous case (2.17) the 10 tensor functions $(f_i)_{mnpq}$ are related to $(\hat f_i)_{mnpq}$ by extracting the scalar phase factor $p_0(\theta,k\,;2)$ (2.15)

$$
f_i(\theta,k)=p_0(\theta,k\,;2)\,\hat f_i(\theta,k)\ . \tag{2.19}
$$

Here the $YYZZ$ and $\zeta\zeta\chi\chi$ amplitudes are not related in the same way as they were in the $AdS_2\times S^2$ case (the functions f_5 and $f_{\bar 5}$ are not equal).

This one-loop S-matrix does not satisfy the Yang-Baxter equation. Similarly to the complex sine-Gordon case [38, 2] one can find a local quartic counterterm whose contribution cancels the underlined terms in the above S-matrix coefficients $\hat f_i$ and thus restores the validity of the Yang-Baxter equation at the one-loop order. Adding this counterterm also restores the equality between the coefficient functions f_5 and $f_{\bar 5}$. This will be discussed in detail in section 4.

2.2.3 Reduced $AdS_5\times S^5$ theory

The one-loop S-matrix computed starting with the Lagrangian (2.3) is

Boson-Boson

$$
\begin{aligned}
\mathbb{S}\,|Y_m(p_1)Y_n(p_2)\rangle=&f_{1mnpq}(\theta,k)\,|Y_p(p_1)Y_q(p_2)\rangle+f_{5mnpq}(\theta,k)\,|Z_p(p_1)Z_q(p_2)\rangle\\
&+f^+_{6\ mnpq}(\theta,k)\,|\zeta_p(p_1)\zeta_q(p_2)\rangle+f^-_{6\ mnpq}(\theta,k)\,|\chi_p(p_1)\chi_q(p_2)\rangle\\
\mathbb{S}\,|Z_m(p_1)Z_n(p_2)\rangle=&f_{1mnpq}(\theta,-k)\,|Z_p(p_1)Z_q(p_2)\rangle+f_{5mnpq}(\theta,-k)\,|Y_p(p_1)Y_q(p_2)\rangle\\
&+f^+_{6\ mnpq}(\theta,-k)\,|\chi_p(p_1)\chi_q(p_2)\rangle+f^-_{6\ mnpq}(\theta,-k)\,|\zeta_p(p_1)\zeta_q(p_2)\rangle\\
\mathbb{S}\,|Y_m(p_1)Z_n(p_2)\rangle=&f_{3mnpq}(\theta,k)\,|Y_p(p_1)Z_q(p_2)\rangle+f_{5mqpn}(i\pi-\theta,k)\,|Z_p(p_1)Y_q(p_2)\rangle\\
&+f^+_{7\ mnpq}(\theta,k)\,|\zeta_p(p_1)\chi_q(p_2)\rangle-f^-_{7mnpq}(\theta,k)\,|\chi_p(p_1)\zeta_q(p_2)\rangle
\end{aligned}
$$

Boson-Fermion

$$
\begin{aligned}
\mathbb{S}\,|Y_m(p_1)\zeta_n(p_2)\rangle=&f^+_{4\ mnpq}(\theta,k)\,|Y_p(p_1)\zeta_q(p_2)\rangle+if^+_{6\ mqpn}(i\pi-\theta,k)\,|\zeta_p(p_1)Y_q(p_2)\rangle\\
&+f^+_{8\ mnpq}(\theta,k)\,|Z_p(p_1)\chi_q(p_2)\rangle+if^-_{7\ mqpn}(i\pi-\theta,k)\,|\chi_p(p_1)Z_q(p_2)\rangle\\
\mathbb{S}\,|Y_m(p_1)\chi_n(p_2)\rangle=&f^-_{4\ mnpq}(\theta,k)\,|Y_p(p_1)\chi_q(p_2)\rangle+if^-_{6\ mqpn}(i\pi-\theta,k)\,|\chi_p(p_1)Y_q(p_2)\rangle\\
&-f^-_{8\ mnpq}(\theta,k)\,|Z_p(p_1)\zeta_q(p_2)\rangle-if^+_{7\ mqpn}(i\pi-\theta,k)\,|\zeta_p(p_1)Z_q(p_2)\rangle\\
\mathbb{S}\,|Z_m(p_1)\zeta_n(p_2)\rangle=&f^-_{4\ mnpq}(\theta,-k)\,|Z_p(p_1)\zeta_q(p_2)\rangle+if^-_{6\ mqpn}(i\pi-\theta,-k)\,|\zeta_p(p_1)Z_q(p_2)\rangle\\
&-f^-_{8\ mnpq}(\theta,-k)\,|Y_p(p_1)\chi_q(p_2)\rangle-if^+_{7\ mqpn}(i\pi-\theta,-k)\,|\chi_p(p_1)Y_q(p_2)\rangle\\
\mathbb{S}\,|Z_m(p_1)\chi_n(p_2)\rangle=&f^+_{4\ mnpq}(\theta,-k)\,|Z_p(p_1)\chi_q(p_2)\rangle+if^+_{6\ mqpn}(i\pi-\theta,-k)\,|\chi_p(p_1)Z_q(p_2)\rangle
\end{aligned}
$$

$$+ f^{+}_{8\,mnpq}(\theta,-k)\,|Y_p(p_1)\zeta_q(p_2)\rangle + i f^{-}_{7\,mqpn}(i\pi-\theta,-k)\,|\zeta_p(p_1)Y_q(p_2)\rangle$$

Fermion-Fermion

$$\begin{aligned}
\mathbb{S}\,|\zeta_m(p_1)\zeta_n(p_2)\rangle =& f_{2mnpq}(\theta,k)\,|\zeta_p(p_1)\zeta_q(p_2)\rangle - f_{5mnpq}(\theta,k)\,|\chi_p(p_1)\chi_q(p_2)\rangle \\
&+ f^{+}_{6\,mnpq}(\theta,k)\,|Y_p(p_1)Y_q(p_2)\rangle + f^{-}_{6\,mnpq}(\theta,-k)\,|Z_p(p_1)Z_q(p_2)\rangle \\
\mathbb{S}\,|\chi_m(p_1)\chi_n(p_2)\rangle =& f_{2mnpq}(\theta,-k)\,|\chi_p(p_1)\chi_q(p_2)\rangle - f_{5mnpq}(\theta,-k)\,|\zeta_p(p_1)\zeta_q(p_2)\rangle \\
&+ f^{+}_{6\,mnpq}(\theta,-k)\,|Z_p(p_1)Z_q(p_2)\rangle + f^{-}_{6\,mnpq}(\theta,k)\,|Y_p(p_1)Y_q(p_2)\rangle \\
\mathbb{S}\,|\zeta_m(p_1)\chi_n(p_2)\rangle =& f_{3mnpq}(\theta,k)\,|\zeta_p(p_1)\chi_q(p_2)\rangle - f_{5mqpn}(i\pi-\theta,k)\,|\chi_p(p_1)\zeta_q(p_2)\rangle \\
&+ f^{+}_{7\,mnpq}(\theta,k)\,|Y_p(p_1)Z_q(p_2)\rangle + f^{-}_{7\,mnpq}(\theta,k)\,|Z_p(p_1)Y_q(p_2)\rangle
\end{aligned}$$

Functions

$$\begin{aligned}
\hat f_{1mnpq}(\theta,k) =& \delta_{mp}\delta_{nq}\big(1 - \frac{i\pi}{k}\,\mathrm{cosech}\,\theta - \frac{\pi^2}{2k^2}\big) \\
&+ \delta_{mn}\delta_{pq}\big(\frac{i\pi}{k}\coth\theta + \frac{\pi}{k^2}(i(i\pi-\theta) - \frac{\pi}{2}(\,\mathrm{cosech}\,\theta - \coth\theta)\,\mathrm{cosech}\,\theta)\big) \\
&+ \delta_{mq}\delta_{np}\big(-\frac{i\pi}{k}\coth\theta + \frac{\pi}{k^2}(i\theta - \frac{\pi}{2}(\,\mathrm{cosech}\,\theta + \coth\theta)\,\mathrm{cosech}\,\theta)\big) + \mathcal{O}(k^{-3})\ , \\
\hat f_{2mnpq}(\theta,k) =& \delta_{mp}\delta_{nq}\big(1 - \frac{\pi^2}{k^2}\big) - \frac{i\pi}{k}\epsilon_{mnpq}\coth\theta \\
&+ \delta_{mn}\delta_{pq}\big(\frac{\pi}{k^2}(-i\theta + \frac{\pi}{2}(\,\mathrm{cosech}\,\theta - \coth\theta)\,\mathrm{cosech}\,\theta)\big) \\
&+ \delta_{mq}\delta_{np}\big(\frac{\pi}{k^2}(-i(i\pi-\theta) + \frac{\pi}{2}(\,\mathrm{cosech}\,\theta + \coth\theta)\,\mathrm{cosech}\,\theta)\big) + \mathcal{O}(k^{-3})\ , \\
\hat f_{3mnpq}(\theta,k) =& \delta_{mp}\delta_{nq} + \mathcal{O}(k^{-3})\ , \\
\hat f^{\pm}_{4\,mnpq}(\theta,k) =& \delta_{mp}\delta_{nq}\big(1 - \frac{i\pi}{2k}\,\mathrm{cosech}\,\theta - \frac{3\pi^2}{8k^2}\big) \\
&+ (\delta_{mn}\delta_{pq} - \delta_{mq}\delta_{np} \mp \epsilon_{mnpq})\big(\frac{i\pi}{2k}\coth\theta + \frac{i\pi}{4k^2}(i\pi - 2\theta)\big) + \mathcal{O}(k^{-3})\ , \\
\hat f_{5mnpq}(\theta,k) =& \frac{\pi^2}{4k^2}\delta_{mn}\delta_{pq}\,\mathrm{sech}^{\,2}\frac{\theta}{2} + \mathcal{O}(k^{-3})\ , \\
\hat f^{\pm}_{6\,mnpq}(\theta,k) =& -\frac{\pi^2}{4k^2}\delta_{mn}\delta_{pq}\,\mathrm{cosech}\,\frac{\theta}{2}(1 + \tanh^2\frac{\theta}{2}) \\
&+ (\delta_{mn}\delta_{pq} + \delta_{mp}\delta_{nq} - \delta_{mq}\delta_{np} \pm \epsilon_{mnpq})\big(\frac{i\pi}{4k}\,\mathrm{sech}\,\frac{\theta}{2} + \frac{\pi^2}{8k^2}\,\mathrm{cosech}\,\frac{\theta}{2}\big) + \mathcal{O}(k^{-3})\ , \\
\hat f^{\pm}_{7\,mnpq}(\theta,k) =& \frac{i\pi}{4k}\,\mathrm{cosech}\,\frac{\theta}{2}\big(\delta_{mp}\delta_{nq} + \delta_{mq}\delta_{np} - \delta_{mn}\delta_{pq} \mp \epsilon_{mnpq}\big) + \mathcal{O}(k^{-3})\ , \\
\hat f^{\pm}_{8\,mnpq}(\theta,k) =& \frac{\pi^2}{4k^2}\,\mathrm{cosech}\,\theta\big(\delta_{mn}\delta_{pq} + \delta_{mq}\delta_{np} - \delta_{mp}\delta_{nq} \pm \epsilon_{mnpq}\big) + \mathcal{O}(k^{-3})\ ,
\end{aligned}$$

Here again we extracted the phase factor $p_0(\theta,k\,;1)$ defined in (2.15), i.e. all f_i in the S-matrix are given in terms of the corresponding $\hat f_i$ by

$$f_i(\theta,k) = p_0(\theta,k\,;1)\,\hat f_i(\theta,k)\ . \tag{2.20}$$

The $1/k$ terms in f_i are tree-level contributions found in [1] while $1/k^2$ terms are new one-loop contributions computed here.

For the reduced $AdS_5\times S^5$ theory the $YYZZ$ and $\zeta\zeta\chi\chi$ amplitudes are related in the same way as they were for the reduced $AdS_2\times S^2$ theory, i.e. in contrast to the $AdS_3\times S^3$ case, here $f_5 = \tilde f_5$. This equality could be related to the group factorisation property of the perturbative S-matrix [2] and may be suggesting the presence of a hidden fermionic symmetry relating bosons and fermions (see also below).

3 S-matrix of the reduced $AdS_2\times S^2$ theory

In [3] the reduced version of the $AdS_2\times S^2$ superstring model based on the $\frac{\hat F}{G} = \frac{PSU(1,1|2)}{SO(1,1)\times SO(2)}$ supercoset was shown to be equivalent to the $\mathcal{N}=2$ supersymmetric sine-Gordon model, whose exact S-matrix is known,

[45, 46, 43, 44]. In this section we review this S-matrix and check that its perturbative expansion indeed matches the one-loop result found in section 2.2.1.

We also identify certain key features of this theory that will be useful in analysing the reduced $AdS_3 \times S^3$ and $AdS_5 \times S^5$ theories. In particular, there is a specific phase factor that also plays a rôle in the reduced $AdS_5 \times S^5$ theory.

3.1 Symmetries

Since in the reduced $AdS_2 \times S^2$ theory $G = SO(1,1) \times SO(2)$, the gauge group H is trivial and therefore the theory has no manifest bosonic symmetry (gauge or global), other than the usual 2-d Poincaré symmetry. Parametrising the group field g in terms of an algebra-valued field (2.1) and expanding out the Lagrangian (1.1) in components we get

$$\begin{aligned} L_2 \;\; = \;\; & \frac{k}{4\pi}\Big[\partial_+\phi\partial_-\phi + \partial_+\varphi\partial_-\varphi + \frac{\mu^2}{2}\left(\cos 2\varphi - \cosh 2\phi\right) + i\,\alpha\partial_-\alpha + i\,\delta\partial_-\delta + i\,\nu\partial_+\nu + i\,\rho\partial_+\rho \\ & -2i\mu\Big(\cosh\phi\cos\varphi\,(\nu\delta + \rho\alpha) + \sinh\phi\sin\varphi(-\rho\delta + \nu\alpha)\Big)\Big]\,. \end{aligned} \tag{3.1}$$

Here ϕ, φ are real bosonic fields and α, δ, ν, ρ are real (hermitian) fermions. Expanding this Lagrangian to quartic order one finds agreement with (2.16) up to simple field and coupling constant redefinitions.

Furthermore, this Lagrangian is exactly that of $\mathcal{N} = 2$ supersymmetric sine-Gordon theory [3], i.e. this theory has $\mathcal{N} = 2$ 2-d worldsheet supersymmetry. The $\mathcal{N} = 2$ supersymmetry algebra can be represented in the following way [10]

$$\mathfrak{so}(1,1) \Subset [\mathfrak{psu}(1|1)]^2 \ltimes \mathbb{R}^2\,. \tag{3.2}$$

The superalgebra $\mathfrak{psu}(1|1)$ has two anticommuting fermionic generators and no bosonic generators. The $\mathbb{R}^2$ central extensions correspond to the light-cone combinations $\mathfrak{P}_\pm$ of the 2-momentum components, i.e. the commutation relations can be written as

$$\{\mathfrak{Q}^i_R,\, \mathfrak{Q}^j_L\} = 0\,, \qquad \{\mathfrak{Q}^i_R,\, \mathfrak{Q}^j_R\} = \delta^{ij}\mathfrak{P}_+\,, \qquad \{\mathfrak{Q}^i_L,\, \mathfrak{Q}^j_L\} = \delta^{ij}\mathfrak{P}_-\,, \qquad i,\, j = 1, 2\,. \tag{3.3}$$

The origin of this $\mathcal{N} = 2$ 2-d supersymmetry in the $AdS_2 \times S^2$ reduced theory appears to be in the global target space supergroup used in the construction of the $AdS_2 \times S^2$ superstring theory as a supercoset GS sigma model. In particular, in the Pohlmeyer reduction [3] the fermionic fields were redefined in such a way that they became charged under the 2-d Lorentz symmetry of the reduced theory (the original GS fermions were 2-d scalars).

We shall assume the scattering states of the theory to be eigenstates of the momentum operator. As the direct-sum superalgebra $[\mathfrak{psu}(1|1)]^2$ commutes with the momentum generators we expect the scattering states at fixed momenta to transform in a bi-representation of this direct sum. Furthermore, as we are dealing with an integrable theory we expect the S-matrix to factorise under the corresponding direct-product symmetry structure [43, 44].

3.2 Group factorisation of the S-matrix

To confirm that the one-loop S-matrix of section 2.2.1 agrees with the perturbative expansion of the exact $\mathcal{N} = 2$ supersymmetric sine-Gordon S-matrix [43, 44] we relabel our states as follows

$$\begin{aligned} |Z\rangle &= |\Phi_{00}\rangle\,, \qquad & |\chi\rangle &= |\Phi_{01}\rangle\,, \\ |Y\rangle &= |\Phi_{11}\rangle\,, \qquad & |\zeta\rangle &= |\Phi_{10}\rangle\,. \end{aligned} \tag{3.4}$$

The $\mathcal{N} = 2$ supersymmetry can be understood as two anticommuting (up to central extensions) $\mathcal{N} = 1$ supersymmetries that act on different indices of $|\Phi_{a\alpha}\rangle$ ($a,\, \alpha, \ldots = 0, 1$). We also take 0 to be a bosonic index and 1 to be a fermionic index, so that the gradings are

$$[0] = 0\,, \qquad [1] = 1\,. \tag{3.5}$$

[10] Our notation for semi-direct sums and central extensions is defined in footnote 1. Also, $[\mathfrak{psu}(1|1)]^2$ stands for $\mathfrak{psu}(1|1) \oplus \mathfrak{psu}(1|1)$.

The S-matrix can then be parametrised in the following way

$$\mathbb{S}\,|\Phi_{a\alpha}\Phi_{b\beta}\rangle = S^{cd,\gamma\delta}_{ab,\alpha\beta}(\theta,k)\,|\Phi_{c\gamma}\Phi_{d\delta}\rangle\ . \tag{3.6}$$

As the reduced $AdS_2 \times S^2$ theory is integrable we expect (as discussed in section 3.1) the S-matrix to factorise under the direct-product symmetry group structure (3.2) as follows

$$S^{c\gamma,d\delta}_{a\alpha,b\beta}(\theta,k) = (-1)^{[\alpha][b]+[\gamma][d]} S_B(\theta,k) S^{cd}_{ab}(\theta,k) S^{\gamma\delta}_{\alpha\beta}(\theta,k)\,. \tag{3.7}$$

Here, following [44], an overall bosonic factor $S_B(\theta,k)$ has been extracted. The $\mathcal{N}=2$ supersymmetric sine-Gordon S-matrix can be understood as a supersymmetrisation of the corresponding bosonic sine-Gordon S-matrix [44, 46]. Therefore, we take this factor to be the same as the sine-Gordon perturbative excitation S-matrix.[11] Its expansion to the one-loop order is

$$S_B(\theta,k) = 1 + \frac{2i\pi}{k}\,\mathrm{cosech}\,\theta - \frac{2\pi^2}{k^2}\,\mathrm{cosech}^{\,2}\theta + \mathcal{O}(k^{-3})\,. \tag{3.8}$$

It is useful also to represent the factorised S-matrix (3.7) as acting on a single field Φ_a as

$$\mathbb{S}\,|\Phi_a(p_1)\Phi_b(p_2)\rangle = S^{cd}_{ab}(\theta,k)\,|\Phi_c(p_1)\Phi_d(p_2)\rangle\ , \tag{3.9}$$

where Φ_0 is a bosonic and Φ_1 is a fermionic state (cf. (3.5)). The one-loop S-matrix of section 2.2.1 has the factorised structure (3.6). Taking into account the bosonic factor (3.8), the one-loop amplitudes of the factorised S-matrix are then given by

$$S^{cd}_{ab}(\theta,k) = \ \tilde{p}_0(\theta,k)\ \hat{S}^{cd}_{ab}(\theta,k)\ , \tag{3.10}$$

where

$$\begin{aligned}
&\hat{S}^{00}_{00}(\theta,k) = 1 + \frac{i\pi}{k}\,\mathrm{cosech}\,\theta + \mathcal{O}(k^{-3})\,, && \hat{S}^{11}_{11}(\theta,k) = 1 - \frac{i\pi}{k}\,\mathrm{cosech}\,\theta + \mathcal{O}(k^{-3})\,, \\
&\hat{S}^{11}_{00}(\theta,k) = -\frac{i\pi}{2k}\,\mathrm{sech}\,\frac{\theta}{2} + \mathcal{O}(k^{-3})\,, && \hat{S}^{00}_{11}(\theta,k) = -\frac{i\pi}{2k}\,\mathrm{sech}\,\frac{\theta}{2} + \mathcal{O}(k^{-3})\,, \\
&\hat{S}^{01}_{01}(\theta,k) = 1 + \mathcal{O}(k^{-3})\,, && \hat{S}^{10}_{01}(\theta,k) = \frac{i\pi}{2k}\,\mathrm{cosech}\,\frac{\theta}{2} + \mathcal{O}(k^{-3})\,.
\end{aligned} \tag{3.11}$$

The overall factor that was extracted (see (3.7))

$$\tilde{p}_0(\theta,k) = 1 - \frac{i\pi}{k}\,\mathrm{cosech}\,\theta + \frac{\pi\,\mathrm{cosech}\,\theta}{4k^2}\big(i(2+(i\pi-2\theta)\coth\theta) - 3\pi\,\mathrm{cosech}\,\theta\big) \tag{3.12}$$

satisfies the following equation

$$S_B(\theta,k)\ \big[\tilde{p}_0(\theta,k)\big]^2\ = p_0(\theta,k\,;1) + \mathcal{O}(k^{-3})\,, \tag{3.13}$$

i.e. it matches the phase factor in section 2.2.1.

Note that the choice of the phase factor (3.12) implies that $\hat{S}^{01}_{01}(\theta,k) = 1$ at the one-loop order. This structure continues to hold to all orders, i.e. translating the factorised form (3.7) back to the original notation of section 2.2.1 we conclude that the $YZ \to YZ$ and $\zeta\chi \to \zeta\chi$ amplitudes are precisely equal to

$$S_B(\theta,k)\ \big[S^{01}_{01}(\theta,k)\big]^2\,. \tag{3.14}$$

This is thus a natural choice for the phase factor and it will be useful also in case of the reduced $AdS_5 \times S^5$ theory discussed below.

[11] The exact S-matrix for the sine-Gordon perturbative excitation is given by

$$S_{sg}(\theta,\Delta) = \frac{\sinh\theta + i\sin\Delta}{\sinh\theta - i\sin\Delta}\,,$$

where Δ is a function of the coupling k. In $\mathcal{N}=2$ supersymmetric sine-Gordon this function is $\Delta = \frac{\pi}{k}$, see [43, 44] and below.

It can be checked that the one-loop S-matrix (3.9) and (3.11) satisfies the Yang-Baxter equation to one-loop order. In terms of the tensor $S_{ab}^{cd}(\theta,k)$ the YBE can be written as

$$\sum_{g,h,j=0}^{1}\Big[(-1)^{[h][g]+[d][j]+[e][f]}S_{ab}^{hg}(\theta_{12},k)S_{hc}^{dj}(\theta_{13},k)S_{gj}^{ef}(\theta_{23},k) \\ -(-1)^{[h][f]+[g][j]+[d][e]}S_{bc}^{gj}(\theta_{23},k)S_{aj}^{hf}(\theta_{13},k)S_{hg}^{de}(\theta_{12},k)\Big]=0\,. \tag{3.15}$$

As an immediate consequence, the graded tensor product of two copies of the S-matrix (3.7) will also satisfy the YBE. Therefore, the one-loop S-matrix computed in section 2.2.1 satisfies the YBE as expected.

3.3 $\mathcal{N}=1$ supersymmetric sine-Gordon S-matrix

In the next two subsections we shall review the construction of the exact $\mathcal{N}=2$ supersymmetric sine-Gordon S-matrix [44]. The first step is to find the $\mathcal{N}=1$ supersymmetric sine-Gordon S-matrix [46, 45]. The $\mathcal{N}=1$ supersymmetric sine-Gordon Lagrangian describes one bosonic and one fermionic degree of freedom. We will be interested in the S-matrix of these excitations [12] that can be denoted as

$$|\phi\rangle \quad \text{and} \quad |\psi\rangle\ . \tag{3.16}$$

The $\mathcal{N}=1$ supersymmetry transforms $\phi\leftrightarrow\psi$.

The S-matrix for this theory was first constructed in [45] where it was diagonalised using the following change of basis of two-particle states

$$\begin{aligned}|S\rangle &= \frac{1}{\sqrt{\cosh\frac{\theta}{2}}}\Big(\sinh\frac{\theta}{4}\,|\phi(p_1)\phi(p_2)\rangle+\cosh\frac{\theta}{4}\,|\psi(p_1)\psi(p_2)\rangle\Big)\,,\\ |T\rangle &= \frac{1}{\sqrt{\cosh\frac{\theta}{2}}}\Big(\cosh\frac{\theta}{4}\,|\phi(p_1)\phi(p_2)\rangle-\sinh\frac{\theta}{4}\,|\psi(p_1)\psi(p_2)\rangle\Big)\,,\\ |U\rangle\,,|V\rangle &= \frac{1}{\sqrt{2}}\Big(\,|\phi(p_1)\psi(p_2)\rangle\mp|\psi(p_1)\phi(p_2)\rangle\Big)\,.\end{aligned} \tag{3.17}$$

The diagonalisation is a consequence of the S-matrix commuting with supersymmetry. This further constrains the S-matrix by demanding that there are only two independent amplitudes

$$\begin{aligned}\mathbb{S}\,|S\rangle &= S_{sg}(\theta,\Delta)\;F_+(\theta,\Delta)\,|S\rangle\ , \qquad & \mathbb{S}\,|T\rangle &= S_{sg}(\theta,\Delta)\;F_-(\theta,\Delta)\,|T\rangle\ ,\\ \mathbb{S}\,|U\rangle &= S_{sg}(\theta,\Delta)\;F_+(\theta,\Delta)\,|U\rangle\ , \qquad & \mathbb{S}\,|V\rangle &= S_{sg}(\theta,\Delta)\;F_-(\theta,\Delta)\,|V\rangle\ ,\end{aligned} \tag{3.18}$$

where Δ is a function of the coupling k. The exact form of this function depends on the particular theory. For example, in both the bosonic sine-Gordon and the $\mathcal{N}=1$ supersymmetric sine-Gordon cases k receives a finite shift. In $\mathcal{N}=2$ supersymmetric sine-Gordon case there is no such shift. In contrast to [45] we have extracted an overall factor $S_{sg}(\theta,\Delta)$ which is the S-matrix for the sine-Gordon perturbative excitation

$$S_{sg}(\theta,\Delta)=\frac{\sinh\theta+i\sin\Delta}{\sinh\theta-i\sin\Delta}\,. \tag{3.19}$$

The $\mathcal{N}=1$ supersymmetric sine-Gordon is an integrable theory and the Yang-Baxter equation (3.15) should be satisfied. This further constrains the S-matrix by requiring that $F_\pm$ are related as [45]

$$F_\pm(\theta,\Delta)=\Big(1\mp\frac{i\sin\frac{\Delta}{2}}{\sinh\frac{\theta}{2}}\Big)R(\theta,\Delta)\,. \tag{3.20}$$

[12] Analogously to the bosonic sine-Gordon theory, here the perturbative excitations of the fields in the Lagrangian correspond not to the elementary excitations, which here are solitons, but to (a limit of) the bound states of the solitons. This is also the case in the $\mathcal{N}=2$ supersymmetric sine-Gordon theory. A brief summary of other sectors of the S-matrix is given in appendix A. For further details see [46, 43].

The common factor $R(\theta,\Delta)$ can be further constrained by using the unitarity and crossing relations [45] [13]

$$\begin{aligned}
R(\theta,\Delta) =&\frac{\sinh\theta - i\sin\Delta}{\sinh\theta+i\sin\Delta}\;Y(\theta,\Delta)\;Y(i\pi-\theta,\Delta)\,,\\
Y(\theta,\Delta) =& \prod_{l=1}^{\infty}\frac{\Gamma\big(\frac{\Delta}{2\pi}-\frac{i\theta}{2\pi}+l\big)\Gamma\big(-\frac{\Delta}{2\pi}-\frac{i\theta}{2\pi}+l-1\big)\Gamma\big(-\frac{i\theta}{2\pi}+l-\frac{1}{2}\big)\Gamma\big(-\frac{i\theta}{2\pi}+l+\frac{1}{2}\big)}{\Gamma\big(\frac{\Delta}{2\pi}-\frac{i\theta}{2\pi}+l+\frac{1}{2}\big)\Gamma\big(-\frac{\Delta}{2\pi}-\frac{i\theta}{2\pi}+l-\frac{1}{2}\big)\Gamma\big(-\frac{i\theta}{2\pi}+l-1\big)\Gamma\big(-\frac{i\theta}{2\pi}+l\big)}\,.
\end{aligned} \tag{3.21}$$

Let us translate this exact S-matrix into the original basis of two-particle states (3.16), again extracting an overall bosonic factor

$$\mathbb{S}\,|\Phi_a(p_1)\Phi_b(p_2)\rangle = \; S_{sg}(\theta,\Delta)\;S_{\mathcal{N}_1}{}_{ab}^{cd}(\theta,\Delta)\,|\Phi_c(p_1)\Phi_d(p_2)\rangle\;. \tag{3.22}$$

Here $\Phi_0=\phi$ and $\Phi_1=\psi$, i.e. 0 is a bosonic and 1 is a fermionic index and the components of $S_{\mathcal{N}_1}(\theta,\Delta)$ are

$$\begin{aligned}
S_{\mathcal{N}_1}{}_{00}^{00}(\theta,\Delta) &= R(\theta,\Delta)\;(1+2i\sin\frac{\Delta}{2}\,\mathrm{cosech}\,\theta)\;, & S_{\mathcal{N}_1}{}_{00}^{11}(\theta,\Delta) &= -\,iR(\theta,\Delta)\;\sin\frac{\Delta}{2}\,\mathrm{sech}\,\frac{\theta}{2}\;,\\
S_{\mathcal{N}_1}{}_{11}^{00}(\theta,\Delta) &= R(\theta,\Delta)\;(1-2i\sin\frac{\Delta}{2}\,\mathrm{cosech}\,\theta)\;, & S_{\mathcal{N}_1}{}_{11}^{11}(\theta,\Delta) &= -\,iR(\theta,\Delta)\;\sin\frac{\Delta}{2}\,\mathrm{sech}\,\frac{\theta}{2}\;,\\
S_{\mathcal{N}_1}{}_{01}^{01}(\theta,\Delta) &= R(\theta,\Delta)\;, & S_{\mathcal{N}_1}{}_{01}^{10}(\theta,\Delta) &= iR(\theta,\Delta)\;\sin\frac{\Delta}{2}\,\mathrm{cosech}\,\frac{\theta}{2}\;.
\end{aligned} \tag{3.23}$$

Motivated by [46, 44] we may think of $S_{\mathcal{N}_1}(\theta,\Delta)$ as a minimal $\mathcal{N}=1$ supersymmetric integrable S-matrix (denoted as $S^{(1,1)}_{RSG}$ in [46]). The S-matrix for the perturbative excitations of the $\mathcal{N}=1$ supersymmetric sine-Gordon model can then be thought of as the tensor product of the S-matrix for the perturbative excitation of the sine-Gordon theory with this supersymmetric S-matrix. This structure extends also to other sectors of the theory [46, 44], e.g., to the soliton-soliton S-matrix. This is discussed briefly in appendix A.

It will be useful to write down the expansion of $R(\theta,\Delta)$ in (3.21) to the one-loop order [14]

$$R(\theta,\Delta) = 1 - i\Delta\,\mathrm{cosech}\,\theta + \frac{\Delta^2\,\mathrm{cosech}\,\theta}{4\pi}\Big(i\big[2+(i\pi-2\theta)\coth\theta\big]-3\pi\,\mathrm{cosech}\,\theta\Big)+\mathcal{O}(\Delta^3)\,. \tag{3.24}$$

3.4 $\mathcal{N}=2$ supersymmetric sine-Gordon S-matrix

If $\Delta(k)$ is given by [15]

$$k(\Delta)=\frac{\pi}{\Delta}\,,\qquad\qquad \Delta(k)=\frac{\pi}{k}\,, \tag{3.25}$$

i.e. k is not shifted, then the expansion of (3.23) matches the factorised result of the one-loop $AdS_2\times S^2$ computation (3.11). Similarly, the expansions of $R(\theta,\Delta)$ (3.24) and the bosonic factor (3.19) match (3.12) and (3.8) respectively. An immediate consequence is that the one-loop perturbative result of section 2.2.1 takes the form [16]

$$S_{sg}(\theta,\frac{\pi}{k})\otimes S_{\mathcal{N}_1}(\theta,\frac{\pi}{k})\otimes_G S_{\mathcal{N}_1}(\theta,\frac{\pi}{k})\,. \tag{3.26}$$

This agrees with the exact result for the $\mathcal{N}=2$ supersymmetric sine-Gordon S-matrix [44]. As in the $\mathcal{N}=1$ sine-Gordon case, this can be thought of as a supersymmetrisation of the bosonic sine-Gordon S-matrix. The

[13] The unitarity and crossing constraints do not have a unique solution. To choose the correct solution one should use additional arguments related to the pole structure of the S-matrix [45]. Alternatively, one can fix the ambiguity by matching the S-matrix with the result of a perturbative field theory computation of scattering amplitudes.

[14] This can be derived using various polygamma identities. Using this expression it is possible to check that the direct result of the one-loop computation of the S-matrix for the perturbative excitations of $\mathcal{N}=1$ supersymmetric sine-Gordon model agrees with the exact expression [45] given in (3.18), (3.20) and (3.21). The relation between k and Δ for this theory is given by [46]

$$k(\Delta)=\frac{\pi}{\Delta}+\frac{1}{2}\,,\qquad\qquad \Delta(k)=\frac{\pi}{k-\frac{1}{2}}\,.$$

This relation is the $\mathcal{N}=1$ supersymmetric sine-Gordon version of the well-known bosonic sine-Gordon coupling constant renormalisation. The relation to the couplings β and γ used in [46] is as follows $k=\frac{2\pi}{\beta^2}$, $\Delta=\frac{\gamma}{8}$.

[15] Taking the relation to their couplings β and γ, $k=\frac{1}{\beta^2}$, $\Delta=\pi\gamma$, this identification (3.25) agrees with the relation between γ and β given in [43], which was derived from central charge and quantum group arguments.

[16] The symbol $\otimes_G$ denotes the graded tensor product, defined with respect to indices in (3.7).

form of the S-matrix (3.26) can be extended to other sectors, e.g., the soliton-soliton S-matrix (see appendix A).

The exact S-matrix (3.26) is written as a tensor product of the three S-matrices each satisfying the YBE by construction. Therefore, it also satisfies the Yang-Baxter equation.

3.5 Phase factor

In this subsection we identify a phase factor that will be useful in the discussion of the S-matrix of the reduced $AdS_5 \times S^5$ theory. Motivated by the factorised form of the S-matrix, (3.7), (3.26), we consider

$$P(\theta,\Delta) = S_{sg}(\theta,\Delta)\ \left[R(\theta,\Delta)\right]^2 . \tag{3.27}$$

$S_{sg}(\theta,\Delta)$ and $R(\theta,\Delta)$ are given in (3.19) and (3.21) respectively. As explained below (3.12) this phase factor equals the amplitudes for the $YZ \to YZ$ and $\zeta\chi \to \zeta\chi$ processes. This can be easily seen from (3.14) and (3.22), (3.23) with

$$S_B(\theta,k) = S_{sg}(\theta,\frac{\pi}{k}) , \qquad S^{01}_{01}(\theta,k) = R(\theta,\frac{\pi}{k}) . \tag{3.28}$$

For later use let us record the expansions of this phase factor and its square root to the one-loop (Δ^2) order

$$P(\theta,\Delta) = 1 + \frac{\Delta^2}{2\pi}\operatorname{cosech}\theta\big(i(2+(i\pi-2\theta)\coth\theta) - \pi\operatorname{cosech}\theta\big) + \mathcal{O}(\Delta^3) . \tag{3.29}$$

$$\sqrt{P(\theta,\Delta)} = 1 + \frac{\Delta^2}{4\pi}\operatorname{cosech}\theta\big(i(2+(i\pi-2\theta)\coth\theta) - \pi\operatorname{cosech}\theta\big) + \mathcal{O}(\Delta^3) . \tag{3.30}$$

As expected, (3.29) matches the factor that was extracted from the one-loop S-matrix in (2.17).

Finally, we present a few identities that are useful for checking the unitarity and crossing relations

$$\begin{aligned} &R(\theta,\Delta)R(-\theta,\Delta) = \frac{\sinh^2\frac{\theta}{2}}{\sinh^2\frac{\theta}{2}+\sin^2\frac{\Delta}{2}} , \qquad && P(\theta,\Delta)P(-\theta,\Delta) = \Big(\frac{\sinh^2\frac{\theta}{2}}{\sinh^2\frac{\theta}{2}+\sin^2\frac{\Delta}{2}}\Big)^2 , \\ &R(\theta,\Delta) = R(i\pi-\theta,\Delta) , && P(\theta,\Delta) = P(i\pi-\theta,\Delta) . \end{aligned} \tag{3.31}$$

4 S-matrix of the reduced $AdS_3 \times S^3$ theory

In this section we shall investigate the S-matrix of the reduced $AdS_3 \times S^3$ theory using the result of the one-loop computation of section 2.2.2 as an input. By a similar argument to that given in [20] for the reduced $AdS_5 \times S^5$ theory the S-matrix of this theory should also be UV finite. [17]

The one-loop S-matrix found in section 2.2.2 does not satisfy the Yang-Baxter equation. Similarly to the complex sine-Gordon case we will show that the integrability can be restored at the one-loop order by the addition of a local counterterm [38]. As in [38, 28] we take integrability at the quantum level as our guiding principle and assume that such a counterterm should naturally appear in this theory.

In [2] the existence of the required counterterm in the complex sine-Gordon case was understood as a consequence of starting with the gauged WZW formulation (1.1), gauge-fixing and integrating out unphysical fields. At present we do not know how to trace the origin of the counterterm (given below) which is required in the reduced $AdS_3 \times S^3$ theory to a quantum contribution coming from path integral based on the action (1.1). Still, in appendix C this counterterm is shown to originate from a particular functional determinant of an operator acting on algebra-valued fields of (1.1).

As we shall see below, the one-loop S-matrix corrected to include the local counterterm contribution group-factorises and also exhibits a novel quantum-deformed supersymmetry. We propose that an exact S-matrix should be fully constrained by demanding quantum-deformed supersymmetry, the Yang-Baxter equation and group factorisation along with the usual physical requirements of unitarity and crossing.

[17] Indeed, using the expansion (including sextic terms) of the Lagrangian obtained by integrating out $A_\pm$ in the axially gauged theory [5], we have checked the UV finiteness of the one-loop $YY \to YY$ scattering amplitude.

4.1 Bosonic symmetries

The $AdS_3 \times S^3$ superstring sigma model is based on the supercoset

$$\frac{\hat{F}}{G} = \frac{PSU(1,1|2) \times PSU(1,1|2)}{SU(1,1) \times SU(2)}$$

and thus the corresponding reduced theory [5] has $G = SU(1,1) \times SU(2)$ and the gauge group $H = [SO(2)]^2$. One can also reformulate the reduced $AdS_3 \times S^3$ theory such that it has $G = U(1,1) \times U(2)$ and the gauge group $H = [SO(2)]^4$, see appendix B. [18] The symmetry action of one of the extra $SO(2)$s on the physical states is trivial [19] but the action of the other one is non-trivial and thus the non-trivial subgroup of H is $[SO(2)]^3$.

It is a feature of theories with an abelian gauge group H that the Lagrangian (1.1) possesses both an H-gauge symmetry and an additional global H symmetry [9, 11, 33]. The fields on which the global part of the gauge symmetry has a linear action are field redefinitions of the fields on which the global H symmetry has a linear action. Therefore, the physical symmetry of on-shell states is a single copy of H.

The Lagrangian (2.18) is written in the form that has manifest global $SO(2)$ symmetry. To uncover the full bosonic symmetry group we observe that when m, n, p, q are $SO(2)$ vector indices we have

$$\gamma_{mnpq} = \epsilon_{mn}\epsilon_{pq} \,, \tag{4.1}$$

where ϵ_{mn} is the usual antisymmetric $SO(2)$ tensor. We can then immediately see that all but the last line of (2.18) is invariant under four separate $SO(2)$s, each of which only acts on one species of field, (Y, Z, ζ, χ). The last line is invariant when these four $SO(2)$s are identified. There are also two additional $SO(2)$s defined as follows ($\Lambda \in SO(2)$)

$$\begin{array}{llll} Y_m \to \Lambda_{mn} Y_n \,, & Z_m \to Z_n \Lambda_{nm} \,, & \zeta_m \to \zeta_m \,, & \chi_m \to \chi_n \,, \\ Y_m \to Y_n \,, & Z_m \to Z_n \,, & \zeta_m \to \zeta_m \Lambda_{mn} \,, & \chi_m \to \chi_n \Lambda_{nm} \,, \end{array} \tag{4.2}$$

One can check that these are symmetries of (2.18) using the following identity

$$\delta_{mn}(\Lambda^2)_{qp} - \Lambda_{mp}\Lambda_{qn} - \Lambda_{qm}\Lambda_{np} = \delta_{mn}\delta_{pq} - \delta_{mp}\delta_{nq} - \delta_{mq}\delta_{np} \,. \tag{4.3}$$

The three $SO(2)$ symmetries and their action on the fields are thus given by [20]

$$\begin{array}{cccc} & SO(2)_C & SO(2)_B & SO(2)_F \\ Y & \mathbf{2} & \mathbf{2} & 0 \\ Z & \mathbf{2} & -\mathbf{2} & 0 \\ \zeta & \mathbf{2} & 0 & \mathbf{2} \\ \chi & \mathbf{2} & 0 & -\mathbf{2} \end{array} \tag{4.4}$$

Let us digress and demonstrate that, as was claimed in section 2, the truncated Lagrangian (2.18) agrees (up to field redefinitions) with the the Lagrangian obtained by fixing a gauge on g and integrating out $A_\pm$ at the classical level [5] (as the gauge group H is abelian here we choose the axial gauging in (1.1); the resulting Lagrangian can then be expanded about the trivial vacuum)

$$\begin{aligned} L_3 \;=\; & \frac{k}{4\pi}\Big[\partial_+\phi\partial_-\phi + \tanh^2\phi\;\partial_+ v\partial_- v + \partial_+\varphi\partial_-\varphi + \tan^2\varphi\;\partial_+ u\partial_- u + \frac{\mu^2}{2}\left(\cos 2\varphi - \cosh 2\phi\right) \\ & + i\,\alpha\partial_-\alpha + i\,\beta\partial_-\beta + i\,\gamma\partial_-\gamma + i\,\delta\partial_-\delta + i\,\lambda\partial_+\lambda + i\,\nu\partial_+\nu + i\,\rho\partial_+\rho + i\,\sigma\partial_+\sigma \\ & - i\tanh^2\phi\big[\partial_+ v(\lambda\nu - \rho\sigma) - \partial_- v(\alpha\beta - \gamma\delta)\big] + i\tan^2\varphi\big[\partial_+ u(\lambda\nu - \rho\sigma) - \partial_- u(\alpha\beta - \gamma\delta)\big] \\ & + (\sec^2\varphi - \operatorname{sech}^2\phi)(\alpha\beta - \gamma\delta)(\lambda\nu - \rho\sigma) \\ & - 2i\mu\Big(\cosh\phi\cos\varphi(\lambda\gamma + \nu\delta + \rho\alpha + \sigma\beta) \\ & \qquad + \sinh\phi\sin\varphi\big[\cos(v+u)(\rho\delta - \sigma\gamma + \lambda\beta - \nu\alpha) - \sin(v+u)(\lambda\alpha + \nu\beta - \rho\gamma - \sigma\delta)\big]\Big)\Big] \,. \end{aligned} \tag{4.5}$$

[18] The dimensions of both G and H have been increased by two, thus there are no extra physical degrees of freedom in the theory. The extra gauge degrees of freedom decouple from the rest of the theory and therefore can be ignored in the construction of the Lagrangian [5] and in the S-matrix computation.

[19] The symmetry acts non-trivially on the gauge field, allowing one to eliminate the corresponding degree of freedom.

[20] The notation is as follows: if the fields Y_m, Z_m transform in the $\mathbf{2}, -\mathbf{2}$ representations, then they transform as ($\Lambda \in SO(2)$) $Y \to \Lambda Y$, $Z \to Z\Lambda$.

Here ϕ, φ, v, u are real commuting fields and α, β, γ, δ, λ, ν, ρ, σ are real anticommuting fields. The $[SO(2)]^3$ symmetry of (2.18) (summarised in (4.4)) is the global part of the gauge group. When a gauge is fixed on g and $A_\pm$ are integrated out this symmetry is completely broken. In theories with abelian gauge groups (e.g. the complex sine-Gordon model) there is also a global H symmetry in addition to the H gauge symmetry. Here it acts as follows

$$\begin{aligned} u \to u + c_1 + c_2 \,, \qquad & v \to v + c_1 - c_2 \,, \\ \alpha + i\beta \to e^{i(c_1+c_3)}(\alpha + i\beta) \,, \qquad & \rho + i\sigma \to e^{i(c_1+c_3)}(\rho + i\sigma) \,, \\ \delta + i\gamma \to e^{i(c_1-c_3)}(\delta + i\gamma) \,, \qquad & \nu + i\lambda \to e^{i(c_1-c_3)}(\nu + i\lambda) \,, \end{aligned} \tag{4.6}$$

where c_1, c_2 and c_3 are the three symmetry parameters. Expanding (4.5) to quartic order in "radial" directions ϕ, φ and using the following field redefinition

$$\begin{aligned} & Y_1 = \phi \ \cos v \ , \quad Y_2 = \phi \ \sin v \ , \quad Z_1 = \varphi \ \cos u \ , \quad Z_2 = \varphi \ \sin u \ , \\ & (\zeta_{R1}, \ \zeta_{R2}, \ \zeta_{L1}, \ \zeta_{L2}, \ \chi_{R1}, \ \chi_{R2}, \ \chi_{L1}, \ \chi_{L2}) = (\alpha, \beta, \rho, \sigma, \delta, \gamma, \nu, \lambda) \,, \end{aligned} \tag{4.7}$$

we find the agreement with (2.18) up to simple field and coupling constant redefinitions as claimed.

4.2 Quantum counterterms and the Yang-Baxter equation

The one-loop S-matrix of section 2.2.2 does not satisfy the Yang-Baxter equation (2.12). The YBE is related to conservation of hidden symmetry charges. As with any global symmetry that is not manifestly preserved by a quantization procedure one may try to maintain it at the quantum level by adding local counterterms.

This is what happens in a similar bosonic model – the complex sine-Gordon theory (whose quartic expansion is a truncation of (2.18)) – where there exists a local quantum counterterm that restores the satisfaction of the YBE at the one-loop level [38]. Assuming YBE, the exact soliton S-matrix for the complex sine-Gordon model was constructed in [28]. The correct interpretation of the theory was conjectured to be based on the gauged WZW theory (a special case of (1.1)) with the required quantum counterterm possibly appearing as a consequence of starting with this gauged WZW action in the path integral. Indeed, in [2] it was shown that the counterterm required to reproduce the YBE-consistent S-matrix of [28] can indeed be derived in this way.

Here we find local counterterms that similarly restore the satisfaction of the YBE at one-loop for the S-matrix of the reduced $AdS_3 \times S^3$ theory. In appendix C we suggest a functional determinant origin for these counterterms, yet it appears that they cannot be naively interpreted as arising from a gauge-fixing procedure or integrating out unphysical fields in path integral for (1.1). There may still be an alternative Lagrangian formulation of this reduced theory that leads to the required counterterms. As explained in appendix C such Lagrangian would involve unphysical fields in fermionic subspaces of the superalgebra $\hat{\mathfrak{f}}$.

To restore the YBE the underlined terms in the coefficients $\hat{f}_i$ of the one-loop S-matrix of section 2.2.2 need to be cancelled. These terms are of the form that could arise from a set of local quartic counterterms. As well as cancelling the underlined terms one can add the following arbitrary correction to the coefficient functions f_1, f_2, f_3, $f_{\tilde{3}}$, f_4

$$\frac{i\pi}{k^2}\Big[\delta_{mp}\delta_{nq} \ c_1(\theta, k) + \epsilon_{mp}\epsilon_{nq} \ c_2(\theta, k)\Big] \tag{4.8}$$

without affecting the YBE.

By analogy with the reduced $AdS_2 \times S^2$ theory and the complex sine-Gordon model [2] we may propose some assumptions that possible counterterms should satisfy: (i) the counterterms should be second order in derivatives and local; (ii) the function $c_1(\theta, k)$ in (4.8) that may be interpreted as an additional contribution to the phase factor should vanish. As we already have a phase factor (the amplitude of $Y_m Z_n \to Y_m Z_n$ and $\zeta_m \chi_n \to \zeta_m \chi_n$ processes) that fits into a pattern with the reduced $AdS_2 \times S^2$ theory it seems sensible to assume that it is not altered; (iii) the counterterms should factorise into two parts – one transforming under 2-d Lorentz symmetry like ∂_+ and the other like ∂_-, with each part separately invariant under the $[SO(2)]^3$ global symmetry (4.4).

The last requirement is motivated by the origin of the complex sine-Gordon counterterms found in [2]. It implies that there should be no counterterms involving four different "species" (Y, Z, ζ and χ). Consequently, there should be no counterterm-induced shift of the S-matrix proportional to the tree-level S-matrix (such a shift could be reinterpreted as a shift of the coupling k). Also, f_8 coefficient should then remain zero at

one-loop. It turns out that the group factorisation of the S-matrix discussed in section 4.3 implies that f_8 should be identically zero to all orders.

Let us consider the following counterterm which satisfies all of the above requirements [21]

$$\Delta\mathcal{L}_3 = \frac{\pi}{k^2}\epsilon_{mn}\epsilon_{pq}(Y_m\partial_+Y_n+Z_m\partial_+Z_n-i\zeta_{Rm}\zeta_{Rn}-i\chi_{Rm}\chi_{Rn})(Y_p\partial_-Y_q+Z_p\partial_-Z_q-i\zeta_{Lp}\zeta_{Lq}-i\chi_{Lp}\chi_{Rq})\,. \quad (4.9)$$

It produces the following corrections to the functions parametrising the S-matrix:

$$\begin{aligned}
\Delta f_{1mnpq}(\theta,k) =&\frac{i\pi}{k^2}\Big(\epsilon_{mn}\epsilon_{pq}(\operatorname{cosech}\theta-\coth\theta)+\epsilon_{mq}\epsilon_{pn}(\operatorname{cosech}\theta+\coth\theta)-2\epsilon_{mp}\epsilon_{nq}\coth\theta\Big)\,,\\
\Delta f_{2mnpq}(\theta,k) =&-\frac{2i\pi}{k^2}\Big((\epsilon_{mn}\epsilon_{pq}+\epsilon_{mq}\epsilon_{pn})\operatorname{cosech}\theta+\epsilon_{mp}\epsilon_{nq}\coth\theta\Big)\,,\\
\Delta f_{3mnpq}(\theta,k) =&\Delta\tilde{f}_{3mnpq}(\theta,k)=-\frac{2i\pi}{k^2}\epsilon_{mp}\epsilon_{nq}\coth\theta\,,\\
\Delta f_{4mnpq}(\theta,k) =&-\frac{2i\pi}{k^2}\epsilon_{mp}\epsilon_{nq}\coth\theta\,, \qquad\qquad (4.10)\\
\Delta f_{5mnpq}(\theta,k) =&\frac{i\pi}{k^2}\epsilon_{mn}\epsilon_{pq}(\operatorname{cosech}\theta-\coth\theta)\,, \qquad \Delta f_{\tilde{5}mnpq}(\theta,k)=\frac{2i\pi}{k^2}\epsilon_{mn}\epsilon_{pq}\operatorname{cosech}\theta\,,\\
\Delta f_{6mnpq}(\theta,k) =&-\frac{i\pi}{k^2}\epsilon_{mn}\epsilon_{pq}\operatorname{sech}\frac{\theta}{2}\,, \qquad \Delta f_{7mnpq}(\theta,k)=0\,, \qquad \Delta f_{8mnpq}(\theta,k)=0\,.
\end{aligned}$$

The corrected functions $\hat{f}_i$ parametrising the one-loop S-matrix of section 2.2.2 are

$$\begin{aligned}
\hat{f}_{1mnpq}(\theta,k) =&\delta_{mp}\delta_{nq}\Big(1-\frac{2i\pi}{k}\operatorname{cosech}\theta-\frac{2\pi^2}{k^2}\coth^2\theta\Big)\\
&+\epsilon_{mp}\epsilon_{nq}\Big(\frac{2i\pi}{k}\coth\theta-\frac{2i\pi}{k^2}\coth\theta-\frac{i\pi}{k^2}(i\pi-2\theta)\operatorname{cosech}^2\theta+\frac{3\pi^2}{k^2}\coth\theta\operatorname{cosech}\theta\Big)\\
\hat{f}_{2mnpq}(\theta,k) =&\delta_{mp}\delta_{nq}\Big(1+\frac{2\pi^2}{k^2}\operatorname{cosech}^2\theta\Big)\\
&+\epsilon_{mp}\epsilon_{nq}\Big(-\frac{2i\pi}{k^2}\coth\theta-\frac{i\pi}{k^2}(i\pi-2\theta)\operatorname{cosech}^2\theta-\frac{\pi^2}{k^2}\coth\theta\operatorname{cosech}\theta\Big)\\
\hat{f}_{3mnpq}(\theta,k) =&\hat{f}_{\tilde{3}mnpq}(\theta,k)=\delta_{mp}\delta_{nq}+\epsilon_{mp}\epsilon_{nq}\Big(-\frac{2i\pi}{k^2}\coth\theta-\frac{i\pi}{k^2}(i\pi-2\theta)\operatorname{cosech}^2\theta+\frac{\pi^2}{k^2}\coth\theta\operatorname{cosech}\theta\Big)\\
\hat{f}_{4mnpq}(\theta,k) =&\delta_{mp}\delta_{nq}\Big(1-\frac{i\pi}{k}\operatorname{cosech}\theta-\frac{\pi^2}{2k^2}\Big) \qquad\qquad (4.11)\\
&+\epsilon_{mp}\epsilon_{nq}\Big(\frac{i\pi}{k}\coth\theta-\frac{2i\pi}{k^2}\coth\theta-\frac{i\pi}{k^2}(i\pi-2\theta)\operatorname{cosech}^2\theta+\frac{\pi^2}{k^2}\coth\theta\operatorname{cosech}\theta\Big)\\
\hat{f}_{5mnpq}(\theta,k) =&\hat{f}_{\tilde{5}mnpq}(\theta,k)=\frac{\pi^2}{2k^2}\operatorname{sech}^2\frac{\theta}{2}(\delta_{mn}\delta_{pq}+\epsilon_{mn}\epsilon_{pq})\\
\hat{f}_{6mnpq}(\theta,k) =&\Big(\frac{i\pi}{2k}\operatorname{sech}\frac{\theta}{2}-\frac{\pi^2}{2k^2}\operatorname{sech}\frac{\theta}{2}\tanh\frac{\theta}{2}\Big)(\delta_{mn}\delta_{pq}+\epsilon_{mn}\epsilon_{pq})\\
\hat{f}_{7mnpq}(\theta,k) =&\frac{i\pi}{2k}\operatorname{cosech}\frac{\theta}{2}\big(-\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}+\delta_{mq}\delta_{np}\big)\\
\hat{f}_{8mnpq}(\theta,k) =&0
\end{aligned}$$

The addition of the counterterm has restored the relation between the $YYZZ$ and $\zeta\zeta\chi\chi$ amplitudes, i.e. now $f_5=f_{\tilde{5}}$. This may indicate that as well as integrability, a fermionic symmetry relating the bosons and the fermions is restored when the required counterterm is added. Indeed, in section 4.4 this S-matrix will be shown to commute with a quantum-deformed supersymmetry.

One may wonder if the above counterterm can be derived from the path integral for the corresponding gauged WZW-based theory (1.1) as was the case for the complex sine-Gordon model [2]. If we perform a

[21] Note that this counterterm may be written as

$$\frac{\pi}{k^2}\mathcal{J}_+\mathcal{J}_-$$

where $\mathcal{J}_\pm$ is the conserved current associated to the manifest global $SO(2)$ symmetry of the Lagrangian (2.18) (to quadratic order and up to a rescaling of fields).

similar analysis to [2] starting with the reduced $AdS_3 \times S^3$ theory we only get a bosonic counterterm that produces part of the correction to f_1

$$\Delta f_{1mnpq}(\theta, k) = -\frac{2i\pi}{k^2}\Big(\delta_{mn}\delta_{pq}(\operatorname{cosech}\theta + \coth\theta) + \delta_{mq}\delta_{np}(\operatorname{cosech}\theta - \coth\theta)\Big)\,. \tag{4.12}$$

It is possible that there is an alternative way of formulating the Lagrangian (1.1) which treats bosons and fermions on a more equal footing. In this case one may be able to obtain the counterterm (4.9) as a contribution of some functional determinant in the one-loop path integral as in [2]. This is discussed further in appendix C (note that some of the notation in appendix C is defined in section 5). [22]

4.3 Group factorisation of the S-matrix

Having added the above counterterm it is possible to repackage the fields in such a way that the resulting S-matrix factorises under some group structure. Consider the following set of $SO(2)$ transformations which are symmetries of the theory

$$\begin{array}{ccccc} & SO(2)_1 & SO(2)_2 & SO(2)_{\dot 1} & SO(2)_{\dot 2} \\ Y_m & \mathbf{2} & 0 & \mathbf{2} & 0 \\ Z_m & 0 & \mathbf{2} & 0 & \mathbf{2} \\ \zeta_m & \mathbf{2} & 0 & 0 & \mathbf{2} \\ \chi_m & 0 & \mathbf{2} & \mathbf{2} & 0 \end{array} \tag{4.13}$$

Any one of these $SO(2)$ transformations can be rewritten as a combination of the other three, agreeing with the symmetry analysis of section 4.1, where global bosonic symmetry of the theory was shown to be $[SO(2)]^3$.

We relabel the fields in terms of their transformations under the four $SO(2)$s (4.13)

$$Y_{a\dot a}\,, \qquad Z_{\alpha\dot\alpha}\,, \qquad \zeta_{a\dot\alpha}\,, \qquad \chi_{\alpha\dot a}\,, \tag{4.14}$$

where the indices a, α, $\dot a$, $\dot\alpha$ ($a = 1, 2$, $\alpha = 3, 4$) are vector indices of $SO(2)_1$, $SO(2)_2$, $SO(2)_{\dot 1}$, $SO(2)_{\dot 2}$ respectively, with the following fermionic grading

$$[a] = [\dot a] = 0\,, \qquad [\alpha] = [\dot\alpha] = 1\,. \tag{4.15}$$

Taking the fields to be real each has four degrees of freedom, whereas they should only have two. To take care of this we impose the following constraints

$$\begin{aligned} Y_{a\dot a} &= -\epsilon_{ab}\epsilon_{\dot a\dot b}Y_{b\dot b}\,, \qquad & Z_{\alpha\dot\alpha} &= -\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}Z_{\beta\dot\beta}\,, \\ \zeta_{a\dot\alpha} &= -\epsilon_{ab}\epsilon_{\dot\alpha\dot\beta}\zeta_{b\dot\beta}\,, \qquad & \chi_{\alpha\dot a} &= -\epsilon_{\alpha\beta}\epsilon_{\dot a\dot b}\chi_{\beta\dot b}\,. \end{aligned} \tag{4.16}$$

For example,

$$Y_{1\dot 1} = -Y_{2\dot 2}\ \ (=\frac{1}{\sqrt{2}}Y_1) \quad \text{and} \quad Y_{1\dot 2} = Y_{2\dot 1}\ \ (=\frac{1}{\sqrt{2}}Y_2)\,. \tag{4.17}$$

These constraints are necessary for the fields to respect all the symmetries given in (4.13).

The field rearrangement (4.14) allows us to consider the single field

$$\Phi_{A\dot A}\,, \qquad A = (a, \alpha) \ \text{and}\ \dot A = (\dot a, \dot\alpha)\,, \tag{4.18}$$

that encodes all four species of field Y, Z, ζ and χ in the natural way.

The counterterm-corrected one-loop S-matrix for the reduced $AdS_3 \times S^3$ theory then factorises as follows

$$\mathbb{S}\,|\Phi_{A\dot A}(p_1)\Phi_{B\dot B}(p_2)\rangle = (-1)^{[\dot A][B]+[\dot C][D]}S_{AB}^{CD}S_{\dot A\dot B}^{\dot C\dot D}\,|\Phi_{C\dot C}(p_1)\Phi_{D\dot D}(p_2)\rangle \tag{4.19}$$

[22] As discussed in section 3 the reduced $AdS_2 \times S^2$ theory is equivalent to $\mathcal{N} = 2$ supersymmetric sine-Gordon for which the exact S-matrix has been derived [44]. The perturbative computation precisely matches this result, i.e. there should be no additional one-loop corrections from local counterterms. For the reduced $AdS_5 \times S^5$ theory the one-loop S-matrix group factorises. It seems likely that there should be no one-loop counterterm corrections there either, or at least they should respect this group factorisation property. Any Lagrangian formulation of the reduced $AdS_3 \times S^3$ theory that gives the required corrections at the one-loop order should then produce no corrections when applied to the cases of the reduced $AdS_2 \times S^2$ and $AdS_5 \times S^5$ theories. The functional determinant contribution discussed in appendix C satisfies this property.

$$S_{AB}^{CD}=\begin{cases} L_1\delta_{ac}\delta_{bd}+L_2\epsilon_{ac}\epsilon_{bd}\,, \\ L_3\delta_{\alpha\gamma}\delta_{\beta\delta}+L_4\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}\,, \\ L_5\delta_{ac}\delta_{\beta\delta}+L_6\epsilon_{ac}\epsilon_{\beta\delta}\,, \\ L_7\delta_{\alpha\gamma}\delta_{bd}+L_8\epsilon_{\alpha\gamma}\epsilon_{bd}\,, \\ L_9(\delta_{ab}\delta_{\gamma\delta}+\epsilon_{ab}\epsilon_{\gamma\delta})\,, \\ L_{10}(\delta_{\alpha\beta}\delta_{cd}+\epsilon_{\alpha\beta}\epsilon_{cd})\,, \\ L_{11}(\delta_{ad}\delta_{\gamma\beta}+\epsilon_{ad}\epsilon_{\gamma\beta})\,, \\ L_{12}(\delta_{\alpha\delta}\delta_{cb}+\epsilon_{\alpha\delta}\epsilon_{cb})\,, \end{cases} \tag{4.20}$$

where L_i are functions of θ and the coupling k.

It is useful to understand the factorised S-matrix (4.20) as acting on a single field

$$\mathbb{S}\,|\Phi_A(p_1)\Phi_B(p_2)\rangle = S_{AB}^{CD}(\theta,k)\,|\Phi_C(p_1)\Phi_D(p_2)\rangle\ , \qquad \Phi_A=(\phi_a,\psi_\alpha)\,, \tag{4.21}$$

where ϕ_a are bosonic and ψ_α are fermionic. This S-matrix should satisfy the usual physical requirements of unitarity and crossing. Unitarity implies

$$(-1)^{[C][D]+[E][F]}\ S_{AB}^{CD}(\theta,k)\ S_{DC}^{FE}(-\theta,k)=\delta_A^E\delta_B^F\,. \tag{4.22}$$

For crossing symmetry we need to introduce the crossed S-matrix denoted by $\bar{S}_{AB}^{CD}(\theta,k)$ which is identical to $S_{AB}^{CD}(\theta,k)$ in (4.21) except with $(L_9,\,L_{10},\,L_{11},\,L_{12})$ replaced by $i(L_9,\,L_{10},\,L_{11},\,L_{12})$. Crossing symmetry then implies

$$S_{AB}^{CD}(\theta,k)=\sum_{E,F=1}^{4}(-1)^{[A][B]+[C][D]}\ C_B^E\ \bar{S}_{FA}^{EC}(i\pi-\theta)\ C^{-1}{}_F^D\,, \tag{4.23}$$

where C_A^B is defined by

$$\begin{gathered}\mathcal{C}\,|\Phi_A\rangle = C_A^B\,|\Phi_B\rangle\ , \\ \mathcal{C}\,|\phi_1\rangle=-\,|\phi_2\rangle\ , \qquad \mathcal{C}\,|\psi_3\rangle=-\,|\psi_4\rangle\ , \qquad \mathcal{C}\,|\phi_2\rangle=|\phi_1\rangle\ , \qquad \mathcal{C}\,|\psi_4\rangle=|\psi_3\rangle\ ,\end{gathered} \tag{4.24}$$

and similarly for $C^{-1}{}_A^B$. Crossing symmetry requires the following relations between the functions L_i,

$$\begin{gathered}L_1(i\pi-\theta,k)=L_1(\theta,k)\,, \qquad L_2(i\pi-\theta,k)=-L_2(\theta,k)\,, \\ L_5(i\pi-\theta,k)=L_5(\theta,k)\,, \qquad L_6(i\pi-\theta,k)=-L_6(\theta,k)\,, \\ L_9(i\pi-\theta,k)=iL_{11}(\theta,k)\,,\end{gathered} \tag{4.25}$$

and similarly for L_3, L_4, L_7, L_8, L_{10}, L_{12}.

For consistency (for example, between $\mathbb{S}\,|Y_m(p_1)\zeta_n(p_2)\rangle$ and $\mathbb{S}\,|\zeta_m(p_1)Y_n(p_2)\rangle$, see appendix D) the functions L_i should also obey the conjugation relations

$$\begin{gathered}L_i(\theta,k)=L_i^*(-\theta,k)\ , \qquad i=1,2,...,8 \\ L_9(\theta,k)=-L_9^*(-\theta,k)\,, \qquad L_{10}(\theta,k)=-L_{10}^*(-\theta,k)\,, \qquad L_{11}(\theta,k)=L_{12}^*(-\theta,k)\,.\end{gathered} \tag{4.26}$$

The Lagrangian (2.18) has also a $\mathbb{Z}_2$ symmetry

$$Y\leftrightarrow Z\,, \qquad \zeta\leftrightarrow\chi\,, \qquad k\to -k\,, \tag{4.27}$$

implying the following relations between the functions L_i

$$\begin{aligned}L_1(\theta,k)&=L_3(\theta,-k)\,, & L_2(\theta,k)&=L_4(\theta,-k)\,, \\ L_5(\theta,k)&=L_7(\theta,-k)\,, & L_6(\theta,k)&=L_8(\theta,-k)\,, \\ L_9(\theta,k)&=-L_{10}(\theta,-k)\,, & L_{11}(\theta,k)&=-L_{12}(\theta,-k)\,.\end{aligned} \tag{4.28}$$

In appendix D equation (4.20) is expanded and rewritten in the original $SO(2)$ notation to enable comparison to the S-matrix of section 2.2.2 with the corrected functions (4.11). Key features of the corrected one-loop S-matrix of section 4.2 are the equality of the $YYZZ$ and $\zeta\zeta\chi\chi$ amplitudes (i.e. $f_3=f_{\tilde{3}}$ and $f_5=f_{\tilde{5}}$) and the vanishing of the function f_8. These along with other relationships between the parametrising functions

are required for the one-loop S-matrix to factorise as in (4.19), (4.20). The one-loop result gives the following L_i

$$L_i(\theta,k) = p_0(\theta,k\,;1)\ \hat{L}_i(\theta,k)\ , \tag{4.29}$$

where the phase factor $p_0(\theta,k\,;1)$ was defined in (2.15) and

$$\begin{aligned}
&\hat{L}_1(\theta,k) = \hat{L}_3(\theta,-k) = 1 - \frac{i\pi}{k}\operatorname{cosech}\theta - \frac{\pi^2}{2k^2} + \mathcal{O}(k^{-3})\\
&\hat{L}_2(\theta,k) = \hat{L}_4(\theta,-k) = \frac{i\pi}{k}\coth\theta - \frac{i\pi}{k^2}\coth\theta - \frac{i\pi}{2k^2}(i\pi-2\theta)(\operatorname{cosech}\theta)^2 + \frac{\pi^2}{2k^2}\coth\theta\operatorname{cosech}\theta + \mathcal{O}(k^{-3})\\
&\hat{L}_5(\theta,k) = \hat{L}_7(\theta,-k) = 1 + \mathcal{O}(k^{-3})\\
&\hat{L}_6(\theta,k) = \hat{L}_8(\theta,-k) = -\frac{i\pi}{k^2}\coth\theta - \frac{i\pi}{2k^2}(i\pi-2\theta)(\operatorname{cosech}\theta)^2 + \frac{\pi^2}{2k^2}\coth\theta\operatorname{cosech}\theta + \mathcal{O}(k^{-3})\\
&\hat{L}_9(\theta,k) = -\hat{L}_{10}(\theta,-k) = \frac{i\pi}{2k}\operatorname{sech}\frac{\theta}{2} + \mathcal{O}(k^{-3})\,, \qquad \hat{L}_{11}(\theta,k) = -\hat{L}_{12}(\theta,-k) = -\frac{i\pi}{2k}\operatorname{cosech}\frac{\theta}{2} + \mathcal{O}(k^{-3})
\end{aligned} \tag{4.30}$$

The phase factor $p_0(\theta,k\,;1)$ is the one-loop expansion of the square root of the factor that was identified in the one-loop S-matrix of the complete reduced $AdS_3\times S^3$ theory, (2.19): the S-matrix has been factorised into two parts, each of which we take with the same phase factor.

The choice of the phase factor in (4.29) retains the structure $\hat{L}_{5,7} = 1 + \mathcal{O}(k^{-3})$ to the one-loop order. From the expanded S-matrix given in appendix D we see that it is not possible to choose a phase factor such that the amplitudes for $Y_m Z_n \to Y_m Z_n$ and $\zeta_m\chi_n \to \zeta_m\chi_n$ scattering processes and $\hat{L}_{5,7}$ both equal one to all orders. This is different from the case of the reduced $AdS_2\times S^2$ theory (3.14) and suggests that these two theories are not part of the same "family". [23]

As well as satisfying the Yang-Baxter equation (which has the same form as in (3.15)) to the one-loop order, the perturbative S-matrix (4.20), (4.21), (4.30) also satisfies the unitarity, crossing and conjugation relations in equations (4.22), (4.23), (4.25), (4.26).

For the purpose of discussing the quantum-deformed supersymmetry in the next subsection it is useful to rewrite the S-matrix (4.21) in terms of the complex fields [24]

$$\phi_{\mathbf{+}} = \phi_1 + i\phi_2\,, \quad \phi_{\mathbf{-}} = \phi_1 - i\phi_2\,, \quad \psi_{\mathbf{+}} = \psi_3 + i\psi_4\,, \quad \psi_{\mathbf{-}} = \psi_3 - i\psi_4\,. \tag{4.31}$$

The S-matrix (4.21) acting on these fields is

$$\begin{aligned}
\mathbb{S}\,|\phi_{\mathbf{+}}\phi_{\mathbf{+}}\rangle =& (L_1(\theta,k) - L_2(\theta,k))\,|\phi_{\mathbf{+}}\phi_{\mathbf{+}}\rangle\\
\mathbb{S}\,|\phi_{\mathbf{+}}\phi_{\mathbf{-}}\rangle =& (L_1(\theta,k) + L_2(\theta,k))\,|\phi_{\mathbf{+}}\phi_{\mathbf{-}}\rangle + 2L_9(\theta,k)\,|\psi_{\mathbf{+}}\psi_{\mathbf{-}}\rangle\\
\mathbb{S}\,|\psi_{\mathbf{+}}\psi_{\mathbf{+}}\rangle =& (L_3(\theta,k) - L_4(\theta,k))\,|\psi_{\mathbf{+}}\psi_{\mathbf{+}}\rangle\\
\mathbb{S}\,|\psi_{\mathbf{+}}\psi_{\mathbf{-}}\rangle =& (L_3(\theta,k) + L_4(\theta,k))\,|\psi_{\mathbf{+}}\psi_{\mathbf{-}}\rangle + 2L_{10}(\theta,k)\,|\phi_{\mathbf{+}}\phi_{\mathbf{-}}\rangle\\
\mathbb{S}\,|\phi_{\mathbf{+}}\psi_{\mathbf{+}}\rangle =& (L_5(\theta,k) - L_6(\theta,k))\,|\phi_{\mathbf{+}}\psi_{\mathbf{+}}\rangle + 2L_{11}(\theta,k)\,|\psi_{\mathbf{+}}\phi_{\mathbf{+}}\rangle\\
\mathbb{S}\,|\phi_{\mathbf{+}}\psi_{\mathbf{-}}\rangle =& (L_5(\theta,k) + L_6(\theta,k))\,|\phi_{\mathbf{+}}\psi_{\mathbf{-}}\rangle\\
\mathbb{S}\,|\psi_{\mathbf{+}}\phi_{\mathbf{+}}\rangle =& (L_7(\theta,k) - L_8(\theta,k))\,|\psi_{\mathbf{+}}\phi_{\mathbf{+}}\rangle + 2L_{12}(\theta,k)\,|\phi_{\mathbf{+}}\psi_{\mathbf{+}}\rangle\\
\mathbb{S}\,|\psi_{\mathbf{+}}\phi_{\mathbf{-}}\rangle =& (L_7(\theta,k) + L_8(\theta,k))\,|\psi_{\mathbf{+}}\phi_{\mathbf{-}}\rangle
\end{aligned} \tag{4.32}$$

This S-matrix clearly respects the $U(1)\times U(1)$ bosonic symmetry under which $\phi_{\mathbf{+}}$ has charges $(1,0)$ and $\psi_{\mathbf{+}}$ has charges $(0,1)$.

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[23] This is not such a surprise when looking at the supercosets of the corresponding superstring sigma models. For $AdS_2\times S^2$ we have $\hat{F} = PSU(1,1|2)$ whereas for $AdS_3\times S^3$ we have $\hat{F} = PS([U(1,1|2)]^2)$, i.e. a direct product. As discussed in [5], the Pohlmeyer reduction of models with a direct product as the numerator group of the supercoset is somewhat different compared to the Pohlmeyer reduction of models with a single group as the numerator group. The reduced $AdS_5\times S^5$ theory for which $\hat{F} = PSU(2,2|4)$ has a stronger relation to the reduced $AdS_2\times S^2$ theory.

[24] The S-matrix has a manifest $U(1)\times U(1)$ symmetry. The field ϕ is then charged under the first $U(1)$ and ψ under the second. Thus it would be more natural to write $\phi_{\mathbf{\pm 0}} = \phi_1 \pm i\phi_2\,,\ \psi_{\mathbf{0\pm}} = \psi_3 \pm i\psi_4\,.$ This notation is cluttered and we will suppress the 0 index. The global $U(1)$ indices are in bold to distinguish them from the Lorentz light-cone indices.

4.4 Quantum-deformed supersymmetry

In this section the invariance of the factorised one-loop S-matrix (4.20), (4.21), (4.30) under a quantum-deformed supersymmetry is demonstrated.

The reduced $AdS_2 \times S^2$ theory of section 3 has a $\mathcal{N} = 2$ worldsheet supersymmetry [3] and one may expect to find a similar 2-d supersymmetry in larger models [3]. This is suggested also by the integrability of the model implying the existence of conserved fermionic charges [16]. Very recently the existence of a (non-local) on-shell supersymmetry in the theory (1.1) was pointed out in [47, 48] and the off-shell generalisation demonstrated also in [48].

Here we take an alternative approach: the idea is to find supersymmetry as a symmetry of the S-matrix for on-shell states. The supersymmetry we shall find below appears to be quantum-deformed and thus it is not immediately clear how it should act on the off-shell fields present in the Lagrangian.

The classical supersymmetry algebra we shall consider below (denoted as $\mathfrak{s}$) is the maximal sub-superalgebra of $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^2$ such that the bosonic subalgebra is $[\mathfrak{so}(2)]^2 \oplus \mathbb{R}^2$. This is motivated by the fact that the reduced $AdS_3 \times S^3$ theory is a truncation of the reduced $AdS_5 \times S^5$ theory, for which there is a similarity with the quantum-deformed $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ R-matrix [36, 37]. We also take into account that the global bosonic symmetry of (4.32) is $[\mathfrak{so}(2)]^2$.

The generators of the this classical supersymmetry algebra are: two $SO(2)$ generators, denoted $\mathfrak{R}$ and $\mathfrak{L}$; two positive chirality supercharges, $\mathfrak{Q}_{\pm\mp}$;[25] two negative chirality supercharges, $\mathfrak{S}_{\pm\mp}$; two central extension generators $\bar{\mathfrak{P}}_\pm$ (which are related to the light-cone components of the 2-d momenta, cf. (3.3)).[26] The corresponding commutation relations are given by

$$\begin{aligned}
&[\mathfrak{R},\, \mathfrak{R}] = 0\,, && [\mathfrak{L},\, \mathfrak{L}] = 0\,,\\
&[\mathfrak{R},\, \mathfrak{Q}_{\pm\mp}] = \pm i\mathfrak{Q}_{\pm\mp}\,, && [\mathfrak{L},\, \mathfrak{Q}_{\pm\mp}] = \mp i\mathfrak{Q}_{\pm\mp}\,,\\
&[\mathfrak{R},\, \mathfrak{S}_{\pm\mp}] = \pm i\mathfrak{S}_{\pm\mp}\,, && [\mathfrak{L},\, \mathfrak{S}_{\pm\mp}] = \mp i\mathfrak{S}_{\pm\mp}\,,\\
&\{\mathfrak{S}_{\pm\mp},\, \mathfrak{Q}_{\pm\mp}\} = 0\,, && \{\mathfrak{S}_{\pm\mp},\, \mathfrak{Q}_{\mp\pm}\} = \pm\frac{i}{2}(\mathfrak{R}+\mathfrak{L}) = \pm\mathfrak{A}\,,\\
&\{\mathfrak{Q}_{\pm\mp},\, \mathfrak{Q}_{\pm\mp}\} = 0\,, && \{\mathfrak{Q}_{\pm\mp},\, \mathfrak{Q}_{\mp\pm}\} = -\bar{\mathfrak{P}}_+\,,\\
&\{\mathfrak{S}_{\pm\mp},\, \mathfrak{S}_{\pm\mp}\} = 0\,, && \{\mathfrak{S}_{\pm\mp},\, \mathfrak{S}_{\mp\pm}\} = \bar{\mathfrak{P}}_-\,.
\end{aligned} \tag{4.33}$$

The linear combination

$$\mathfrak{A} \equiv \frac{i}{2}(\mathfrak{R}+\mathfrak{L}) \tag{4.34}$$

commutes with all other generators. This superalgebra may be represented as

$$\mathfrak{s} = \mathfrak{t} \ltimes \mathfrak{so}(2) \ltimes \mathbb{R}^2\,, \tag{4.35}$$

where $\mathfrak{so}(2)$ corresponds to $\mathfrak{A}$, $\mathbb{R}^2$ to $\bar{\mathfrak{P}}_\pm$ and the superalgebra $\mathfrak{t}$ has a bosonic subalgebra $\mathfrak{so}(2)$, generated by

$$\mathfrak{B} \equiv \mathfrak{R} - \mathfrak{L}\,. \tag{4.36}$$

The commutation relations for the superalgebra $\mathfrak{t}$ are

$$\begin{gathered}
[\mathfrak{B}, \mathfrak{B}] = 0\,,\\
[\mathfrak{B},\, \mathfrak{Q}_{\pm\mp}] = \pm i\mathfrak{Q}_{\pm\mp}\,, \quad [\mathfrak{B},\, \mathfrak{S}_{\pm\mp}] = \pm i\mathfrak{S}_{\pm\mp}\,,\\
\{\mathfrak{S}_{\pm\mp},\, \mathfrak{Q}_{\pm\mp}\} = 0\,, \quad \{\mathfrak{S}_{\pm\mp},\, \mathfrak{Q}_{\mp\pm}\} = 0\,,\\
\{\mathfrak{Q}_{\pm\mp},\, \mathfrak{Q}_{\pm\mp}\} = 0\,, \quad \{\mathfrak{Q}_{\pm\mp},\, \mathfrak{Q}_{\mp\pm}\} = 0\,,\\
\{\mathfrak{S}_{\pm\mp},\, \mathfrak{S}_{\pm\mp}\} = 0\,, \quad \{\mathfrak{S}_{\pm\mp},\, \mathfrak{S}_{\mp\pm}\} = 0\,.
\end{gathered} \tag{4.37}$$

This superalgebra (which apparently does not have a standard name) is a semi-direct sum of $\mathfrak{so}(2)$ with two

[25] Note that here the labels $+$ and $-$ do not denote the chirality of the supercharges, but rather the charges under the $SO(2) \times SO(2)$ bosonic subalgebra.

[26] The bar in $\bar{\mathfrak{P}}_\pm$ indicates that these generators may have different k-dependent normalization compared to those in (3.3).

copies of the $\mathfrak{psu}(1|1)$ [27]

$$\mathfrak{t} = \mathfrak{so}(2) \Subset \big[\mathfrak{psu}(1|1)\big]^2 . \tag{4.38}$$

We expect $\mathfrak{A}$ to be a central extension because it and its corresponding symmetry acting on the other half of the factorised S-matrix are actually the same symmetry (which can be seen from (4.13)). Similarly to the $\mathbb{R}^2$ central extensions $\bar{\mathfrak{P}}_\pm$ we do not have two copies of this $\mathfrak{so}(2)$ central extension when we consider the symmetry of the full S-matrix (4.19)

$$[\mathfrak{t}]^2 \ltimes \mathfrak{so}(2) \ltimes \mathbb{R}^2 . \tag{4.39}$$

This is also in agreement with the fact that the global bosonic symmetry is $[SO(2)]^3$.

Given that the bosonic subalgebra of $\mathfrak{s}$ defined by (4.37) is abelian, it should not be altered by a quantum deformation (the S-matrix satisfies the Yang-Baxter equation while respecting the classical $SO(2) \times SO(2)$ symmetry). As a result the *quantum deformation* of $\mathfrak{s}$ we are interested in is rather simple. To construct it we replace the anticommutation relation for $\mathfrak{S}_{\pm\mp}$ and $\mathfrak{Q}_{\mp\pm}$ with

$$\{\mathfrak{S}_{\pm\mp}, \mathfrak{Q}_{\mp\pm}\} = \pm\big[\mathfrak{A}\big]_q , \tag{4.40}$$

where q is a quantum deformation parameter and we use the standard notation

$$\big[x\big]_q \equiv \frac{q^x - q^{-x}}{q - q^{-1}} . \tag{4.41}$$

The generators then have the following action on the one-particle states

$$\begin{aligned}
&\mathfrak{R}\,|\phi_\pm\rangle = \pm i\,|\phi_\pm\rangle , && \mathfrak{R}\,|\psi_\pm\rangle = 0 , \\
&\mathfrak{L}\,|\phi_\pm\rangle = 0 , && \mathfrak{L}\,|\psi_\pm\rangle = \pm i\,|\psi_\pm\rangle , \\
&\mathfrak{Q}_{\pm\mp}\,|\phi_\pm\rangle = 0 , && \mathfrak{Q}_{\pm\mp}\,|\psi_\pm\rangle = d(\vartheta,k)\,|\phi_\pm\rangle , \\
&\mathfrak{Q}_{\pm\mp}\,|\phi_\mp\rangle = c(\vartheta,k)\,|\psi_\mp\rangle , && \mathfrak{Q}_{\pm\mp}\,|\psi_\mp\rangle = 0 , \\
&\mathfrak{S}_{\pm\mp}\,|\phi_\pm\rangle = 0 , && \mathfrak{S}_{\pm\mp}\,|\psi_\pm\rangle = b(\vartheta,k)\,|\phi_\pm\rangle , \\
&\mathfrak{S}_{\pm\mp}\,|\phi_\mp\rangle = a(\vartheta,k)\,|\psi_\mp\rangle , && \mathfrak{S}_{\pm\mp}\,|\psi_\mp\rangle = 0 ,
\end{aligned} \tag{4.42}$$

$$\bar{\mathfrak{P}}_\pm\,|\Phi\rangle = \mathcal{P}_\pm(\vartheta,k)\,|\Phi\rangle . \tag{4.43}$$

For the closure of the quantum-deformed supersymmetry algebra we require

$$ab = \mathcal{P}_- , \qquad cd = -\mathcal{P}_+ , \qquad ad = \big[\frac{1}{2}\big]_q , \qquad bc = -\big[\frac{1}{2}\big]_q . \tag{4.44}$$

To consider the action of the quantum-deformed supersymmetry on the two-particle states a coproduct Δ is required (see, e.g., [50]). This coproduct should respect the quantum-deformed (anti-)commutation relations (4.40) [28]

$$\begin{aligned}
\Delta(\mathfrak{R}) = \mathfrak{R}\otimes\mathbb{I} + \mathbb{I}\otimes\mathfrak{R}, \qquad \Delta(\mathfrak{L}) &= \mathfrak{L}\otimes\mathbb{I} + \mathbb{I}\otimes\mathfrak{L} , \\
\Delta(\mathfrak{Q}_{\pm\mp}) &= \mathfrak{Q}_{\pm\mp}\otimes q^{-\mathfrak{A}} + \mathbb{I}\otimes\mathfrak{Q}_{\pm\mp} , \\
\Delta(\mathfrak{S}_{\pm\mp}) &= \mathfrak{S}_{\pm\mp}\otimes\mathbb{I} + q^{\mathfrak{A}}\otimes\mathfrak{S}_{\pm\mp} , \\
\Delta(\bar{\mathfrak{P}}_\pm) &= \bar{\mathfrak{P}}_\pm\otimes\mathbb{I} + \mathbb{I}\otimes\bar{\mathfrak{P}}_\pm .
\end{aligned} \tag{4.45}$$

This coproduct co-commutes with the S-matrix [29]

$$\Delta_{op}(\mathfrak{J})\;\mathbb{S} \;=\; \mathbb{S}\;\Delta(\mathfrak{J}) \tag{4.46}$$

[27] The appearance of this algebra may be expected given the origin of the reduced theory. In the reduced $AdS_2 \times S^2$ model we start with the supercoset $\frac{PSU(1,1|2)}{SO(1,1)\times SO(2)}$ and end up with the symmetry algebra $[\mathfrak{psu}(1|1)]^2 \ltimes \mathbb{R}^2$ (3.2) in the reduced theory. The reduced $AdS_3 \times S^3$ theory arises from the Pohlmeyer reduction of the 2-d sigma model with the target space $\frac{PS([U(1,1|2)]^2)}{U(1,1)\times U(2)}$ (we use notation defined in appendix B). In the $AdS_3 \times S^3$ case we have two copies of $U(1,1|2)$ and thus we may expect part of the symmetry to be given by $[PSU(1|1)]^4$. This is discussed in detail in section 6.2.2.

[28] As was noted above, the linear combination $\mathfrak{A} = \frac{i}{2}(\mathfrak{R} + \mathfrak{L})$ commutes with all the other generators including the fermionic ones.

[29] The opposite coproduct, $\Delta_{\rm op}(\mathfrak{J})$, is obtained by acting with the graded permutation operator for the tensor product on the original coproduct $\Delta(\mathfrak{J})$.

for an appropriate choice of a, b, c, d in (4.42) and q. Assuming that the quantum deformation parameter is related to the coupling k by (see also below)

$$q = \exp(-\frac{2\pi i}{k}) , \tag{4.47}$$

we find

$$a(\vartheta, k) = \sqrt{\frac{1}{2}\sec\frac{\pi}{k}}\; e^{-\frac{\vartheta}{2}-\frac{i\pi}{2k}} , \qquad b(\vartheta, k) = a^*(\vartheta, k) , \qquad c(\vartheta, k) = -e^{\vartheta}\; a(\vartheta, k) , \qquad d(\vartheta, k) = e^{\vartheta}\; a^*(\vartheta, k) . \tag{4.48}$$

Using (4.44) this implies that the eigenvalues of the central charges $\bar{\mathfrak{P}}_\pm$ are

$$\mathcal{P}_\pm(\vartheta, k) = \frac{1}{2}\sec\frac{\pi}{k}\; e^{\pm\vartheta} , \tag{4.49}$$

i.e. $\bar{\mathfrak{P}}_\pm$ can indeed be identified with the lightcone momentum generators up to normalisation. This suggests that the algebra (4.33) is an $\mathcal{N} = 4$ (i.e. $(4,4)$) 2-d supersymmetry with a non-trivial global bosonic R-symmetry subalgebra. The supercharges (whose anticommutator is proportional to the 2-d momentum operator) are charged under both the Lorentz group, and the global bosonic symmetry group. It is the existence of this global bosonic R-symmetry that allows for a quantum deformation of the supersymmetry algebra.

Expanding (4.40) at large k we have

$$\big[\mathfrak{A}\big]_q = \mathfrak{A} + \frac{2\pi^2}{3k^2}(\mathfrak{A} - \mathfrak{A}^3) + \mathcal{O}(k^{-3}) , \tag{4.50}$$

so that the supersymmetry algebra remains standard at the leading tree-level order. This may be related to a recent suggestion about the existence of an on-shell supersymmetry as part of integrable hierarchy in this classical theory in [16] (see also [47, 48]). Note, however, that for large k the non-trivial coproduct in (4.45) differs from the standard one already by $1/k$ terms, for example,

$$\Delta(\mathfrak{Q}_{\pm\mp}) = \mathfrak{Q}_{\pm\mp} \otimes \mathbb{I} + \mathbb{I} \otimes \mathfrak{Q}_{\pm\mp} + \frac{2\pi i}{k}\mathfrak{Q}_{\pm\mp} \otimes \mathfrak{A} + \mathcal{O}(k^{-2}) . \tag{4.51}$$

The $1/k$ terms are required for the tree-level S-matrix (given by the $1/k$ terms in f_i in (4.11)) to be invariant under the undeformed supersymmetry algebra (4.33). In this sense the supersymmetry of the S-matrix is deformed already at the tree level. This is different to the $AdS_2 \times S^2$ case discussed in section 3.

While the reason for a quantum deformation of the supersymmetry of the S-matrix of the reduced $AdS_3 \times S^3$ theory is not completely clear the above simple construction appears to be consistent and suggests that a similar quantum-deformed supersymmetry may also be present in the reduced $AdS_5 \times S^5$ theory. Let us note that only the bosonic symmetries of the algebra $\mathfrak{s}$ are obvious symmetries of the Lagrangian (2.18). As these bosonic symmetries are abelian, they act on the S-matrix with the standard coproduct. The $AdS_5 \times S^5$ case, for which the bosonic symmetries are non-abelian, should be more non-trivial as the coproduct of the bosonic symmetry generators will also be quantum-deformed (see below).

4.5 Exact S-matrix conjecture

Assuming the quantum-deformed supersymmetry discussed in the previous subsection 4.4 exists to all orders in the $1/k$ expansion one can conjecture an exact S-matrix for the perturbative excitations of the reduced $AdS_3 \times S^3$ theory. Co-commutativity of the S-matrix with the quantum group coproduct (4.45) constrains the form of the S-matrix up to two functions $P_1(\theta, k)$, $P_1(\theta, k)$. The most general functions L_i parametrising a relativistic S-matrix (4.21), (4.20), which co-commutes with the quantum-deformed supersymmetry of section 4.4 are given by

$$\begin{aligned}
L_{1,3}(\theta, k) &= \frac{1}{2}\Big[P_1(\theta, k)\cosh\Big(\frac{\theta}{2} \pm \frac{i\pi}{k}\Big)\operatorname{sech}\frac{\theta}{2} + P_2(\theta, k)\sinh\Big(\frac{\theta}{2} \mp \frac{i\pi}{k}\Big)\operatorname{cosech}\frac{\theta}{2}\Big] , \\
L_{2,4}(\theta, k) &= \frac{1}{2}\Big[P_1(\theta, k)\cosh\Big(\frac{\theta}{2} \pm \frac{i\pi}{k}\Big)\operatorname{sech}\frac{\theta}{2} - P_2(\theta, k)\sinh\Big(\frac{\theta}{2} \mp \frac{i\pi}{k}\Big)\operatorname{cosech}\frac{\theta}{2}\Big] , \\
L_{5,7}(\theta, k) &= \frac{1}{2}\Big[P_1(\theta, k) + P_2(\theta, k)\Big] , \qquad L_{6,8}(\theta, k) = \frac{1}{2}\Big[P_1(\theta, k) - P_2(\theta, k)\Big],
\end{aligned} \tag{4.52}$$

$$L_{9,10}(\theta,k)=\frac{i}{2}P_1(\theta,k)\sin\frac{\pi}{k}\,\mathrm{sech}\,\frac{\theta}{2}\,,\qquad L_{11,12}(\theta,k)=-\frac{i}{2}P_2(\theta,k)\sin\frac{\pi}{k}\,\mathrm{cosech}\,\frac{\theta}{2}$$

It can be checked that this S-matrix satisfies the Yang-Baxter equation, which has the same form as in (3.15). The two phase factors P_1, P_2 will be fixed by using the conditions of crossing, unitarity and consistency with the perturbative one-loop result we obtained above.

To match the one-loop S-matrix (4.30) we require

$$\begin{aligned}P_1(\theta,k) &= 1-\frac{i\pi}{2k^2}(\,\mathrm{sech}\,\frac{\theta}{2})^2\;(\theta+\sinh\theta)+\mathcal{O}(\frac{1}{k^4})\,,\\ P_2(\theta,k) &= 1+\frac{i\pi}{2k^2}(\,\mathrm{cosech}\,\frac{\theta}{2})^2\;\big[(i\pi-\theta)+\sinh\theta\big]+\mathcal{O}(\frac{1}{k^4})\,.\end{aligned}\tag{4.53}$$

Crossing symmetry (4.23) and unitarity (4.22) imply the following constraints on the phase factors

$$P_1(i\pi-\theta,k)=P_2(\theta,k)\;,\tag{4.54}$$

$$P_1(\theta,k)P_1(-\theta,k)=1\,,\qquad P_2(\theta,k)P_2(-\theta,k)=\frac{\sinh^2\frac{\theta}{2}}{\sinh^2\frac{\theta}{2}+\sin^2\frac{\pi}{k}}\,.\tag{4.55}$$

The $\mathbb{Z}_2$ symmetry (4.27) implies

$$P_{1,2}(\theta,k)=P_{1,2}(\theta,-k)\,.\tag{4.56}$$

As expected, the perturbative expressions (4.53) satisfy these relations.

To solve (4.55), (4.54) we use the ansatz [51]

$$P_2(\theta,k)=p_2(\theta,k)\prod_{l=1}^{\infty}\frac{\rho(\theta+2i\pi l,k)}{\rho(-\theta+2i\pi(l+1),k)}\,,\tag{4.57}$$

where $\rho(\theta,k)$ is an arbitrary function and we assume that $p_2(\theta,k)$ satisfies the following relations

$$p_2(\theta,k)\;p_2(-\theta,k)=1\,,\qquad p_2(i\pi-\theta,k)\;p_2(i\pi+\theta,k)=1\,.\tag{4.58}$$

The crossing relation (4.54) implies

$$P_1(\theta,k)=p_2(i\pi-\theta,k)\prod_{l=1}^{\infty}\frac{\rho(\theta+2i\pi(l+\frac{1}{2}),k)}{\rho(-\theta+2i\pi(l+\frac{1}{2}),k)}\,.\tag{4.59}$$

The second relation in (4.58) implies that the first equation in (4.55) is satisfied by construction. The second equation in (4.55) implies

$$\rho(\theta+2i\pi,k)\;\rho(-\theta+2i\pi,k)=\frac{\sinh^2\frac{\theta}{2}}{\sinh^2\frac{\theta}{2}+\sin^2\frac{\pi}{k}}=\frac{\sinh^2\frac{\theta}{2}}{\sinh\big(\frac{\theta}{2}+\frac{i\pi}{k}\big)\sinh\big(\frac{\theta}{2}-\frac{i\pi}{k}\big)}\,.\tag{4.60}$$

Using the gamma function reflection formula this equation is solved by [30]

$$\rho(\theta,k)=\frac{\Gamma(-\frac{i\theta}{2\pi}-\frac{1}{k}-1)\Gamma(-\frac{i\theta}{2\pi}+\frac{1}{k})}{\Gamma(-\frac{i\theta}{2\pi}-1)\Gamma(-\frac{i\theta}{2\pi})}\,.\tag{4.61}$$

To fix $p_2(\theta,k)$ we use the relation (4.56), implying

$$p_2(\theta,k)=\frac{\sinh\big(\frac{\theta}{2}-\frac{i\pi}{k}\big)}{\sinh\big(\frac{\theta}{2}+\frac{i\pi}{k}\big)}\;p_2(\theta,-k)\,.\tag{4.62}$$

[30]There are alternative solutions [51]; the obvious ones are gotten by considering $k\to -k$ and $\theta\to-\theta$. As we require that $P_1(\theta,k)=P_1(\theta,-k)$ and $P_2(\theta,k)=P_2(\theta,-k)$ we can ignore the $k\to-k$ solution. We also disregard the $\theta\to-\theta$ solution as its expansion does not match the perturbative result (4.53). For a similar reason we ignore the solution

$$\rho(\theta,k)=\pm\frac{\sinh\frac{\theta}{2}}{\sinh\big(\frac{\theta}{2}\pm\frac{i\pi}{k}\big)}\,.$$

The minimal choice for the function $p_2(\theta,k)$ that satisfies (4.58) and (4.62) is

$$p_2(\theta,k)=\sqrt{\frac{\sinh\left(\frac{\theta}{2}-\frac{i\pi}{k}\right)}{\sinh\left(\frac{\theta}{2}+\frac{i\pi}{k}\right)}}\,. \tag{4.63}$$

We end up with the following solution for the two functions P_1, P_2:

$$\begin{aligned}
P_1(\theta,k)=&\sqrt{\frac{\cosh\left(\frac{\theta}{2}+\frac{i\pi}{k}\right)}{\cosh\left(\frac{\theta}{2}-\frac{i\pi}{k}\right)}}\prod_{l=1}^{\infty}\frac{\Gamma(\frac{i\theta}{2\pi}-\frac{1}{k}+l-\frac{1}{2})\Gamma(\frac{i\theta}{2\pi}+\frac{1}{k}+l+\frac{1}{2})}{\Gamma(-\frac{i\theta}{2\pi}-\frac{1}{k}+l-\frac{1}{2})\Gamma(-\frac{i\theta}{2\pi}+\frac{1}{k}+l+\frac{1}{2})}\,\frac{\Gamma(-\frac{i\theta}{2\pi}+l-\frac{1}{2})\Gamma(-\frac{i\theta}{2\pi}+l+\frac{1}{2})}{\Gamma(\frac{i\theta}{2\pi}+l-\frac{1}{2})\Gamma(\frac{i\theta}{2\pi}+l+\frac{1}{2})}\,,\\
P_2(\theta,k)=&\sqrt{\frac{\sinh\left(\frac{\theta}{2}-\frac{i\pi}{k}\right)}{\sinh\left(\frac{\theta}{2}+\frac{i\pi}{k}\right)}}\prod_{l=1}^{\infty}\frac{\Gamma(-\frac{i\theta}{2\pi}-\frac{1}{k}+l-1)\Gamma(-\frac{i\theta}{2\pi}+\frac{1}{k}+l)}{\Gamma(\frac{i\theta}{2\pi}-\frac{1}{k}+l)\Gamma(\frac{i\theta}{2\pi}+\frac{1}{k}+l+1)}\,\frac{\Gamma(\frac{i\theta}{2\pi}+l)\Gamma(\frac{i\theta}{2\pi}+l+1)}{\Gamma(-\frac{i\theta}{2\pi}+l-1)\Gamma(-\frac{i\theta}{2\pi}+l)}\,.
\end{aligned} \tag{4.64}$$

It can be checked directly that (4.64) matches the perturbative expansions (4.53). We therefore conjecture that the exact S-matrix for the perturbative excitations of the reduced $AdS_3\times S^3$ theory is given by (4.19), (4.20), (4.52) with phase factors (4.64).

Note that translating the factorised form (4.19) back to the original notation of section 2.2.2 neither the exact $Y_mZ_n\to Y_mZ_n$ and $\zeta_m\chi_n\to\zeta_m\chi_n$ amplitudes nor the square of $L_{5,7}$ are equal to (3.27). This indicates that the reduced $AdS_3\times S^3$ theory is not exactly in the same class of models as the reduced $AdS_2\times S^2$ and $AdS_5\times S^5$ theories.

Still, the product of the two phase factors (4.64) is equal to the square root of (3.27) with $\Delta=\frac{2\pi}{k}$. Also, the factors in front of the products of gamma functions in (4.64) are square roots of the amplitudes in the complex sine-Gordon S-matrix, see [28]. This suggests, by analogy with the reduced $AdS_2\times S^2$ theory which may be interpreted as an $\mathcal{N}=2$ supersymmetric dressing of the bosonic sine-Gordon theory, that the reduced $AdS_3\times S^3$ theory may be interpreted as a quantum-deformed $\mathcal{N}=4$ supersymmetric dressing of the complex sine-Gordon model.

5 Symmetries of the S-matrices

In this section we shall discuss and compare various global symmetries of the reduced $AdS_2\times S^2$, $AdS_3\times S^3$ and $AdS_5\times S^5$ theories with a motivation to understand the expected symmetry of the S-matrix of the $AdS_5\times S^5$ case. We shall see that the latter may be related to the quantum-deformed $\mathfrak{psu}(2|2)\ltimes\mathbb{R}^3$ R-matrix of [36, 37].

The reduced $AdS_2\times S^2$ and $AdS_3\times S^3$ theories are invariant under a wide range of different types of symmetries – 2-d Poincaré, global bosonic symmetries, gauge symmetries, classical supersymmetries and quantum-deformed symmetries. The 2-d Poincaré algebra [31]

$$\mathfrak{iso}(1,1)=\mathfrak{so}(1,1)\Subset\mathbb{R}^2\,. \tag{5.1}$$

contains one Lorentz boost, space translation and time translation under which all the reduced $AdS_n\times S^n$ theories are invariant. In the reduced $AdS_2\times S^2$ and $AdS_3\times S^3$ theories the symmetry that acts on the two-particle states and co-commutes with the S-matrix in each case is based on a superalgebra of the form

$$\mathfrak{c}\ltimes\mathbb{R}^2\,. \tag{5.2}$$

Here $\mathbb{R}^2$ corresponds to the lightcone momenta and $\mathfrak{c}$ contains the fermionic generators (charged under the Lorentz group) whose anticommutator is proportional to the momenta. The fermionic generators are also charged under the bosonic subalgebra $\mathfrak{b}$ of $\mathfrak{c}$. The algebra (5.2) thus has the same structure as a 2-d supersymmetry algebra with a bosonic R-symmetry algebra given by $\mathfrak{b}$.

This symmetry appears to originate from the global target space supersymmetry in the associated superstring theory – the algebra $\mathfrak{c}$ is a particular sub-superalgebra of the latter symmetry. Under the reduction procedure the fermionic target space supersymmetry generators become charged under the Lorentz group and behave like generators of 2-d supersymmetry in the reduced theory.

[31] $\Subset$ denotes the semi-direct sum defined in footnote 1.

In the case of the reduced $AdS_2 \times S^2$ theory the bosonic subalgebra $\mathfrak{b}$ is absent and all the fermionic generators of $\mathfrak{c}$ anticommute up to the central extension generators. Thus a quantum deformation of the corresponding algebra of a kind discussed in section 4.4 is trivial here. In both the reduced $AdS_2 \times S^2$ and $AdS_3 \times S^3$ theories we can write the physical symmetry of the corresponding S-matrix as [32]

$$U_q\big(\mathfrak{so}(1,1) \Subset (\mathfrak{c} \ltimes \mathbb{R}^2)\big)\,. \tag{5.3}$$

5.1 Algebraic structure of Pohlmeyer reduction

In order to understand the origin of the superalgebra $\mathfrak{c}$ it is useful to review the algebraic structure of Pohlmeyer reduction [3]. The reduced $AdS_n \times S^n$ theories are Pohlmeyer reductions of the GS superstring sigma model based on a supercoset space $\hat{F}/G$ where $\hat{F}$ is a supergroup and G is some bosonic subgroup. The superalgebra of $\hat{F}$ is required to have a $\mathbb{Z}_4$ decomposition

$$\hat{\mathfrak{f}} = \hat{\mathfrak{f}}_0 + \hat{\mathfrak{f}}_1 + \hat{\mathfrak{f}}_2 + \hat{\mathfrak{f}}_3\,, \qquad [\hat{\mathfrak{f}}_i,\, \hat{\mathfrak{f}}_j] \subset \hat{\mathfrak{f}}_{i+j \,\mathrm{mod}\, 4}\,, \tag{5.4}$$

where even/odd subscripts denote bosonic/fermionic subspaces. The algebra $\mathfrak{g}$ of the group G is identified as

$$\mathfrak{g} = \hat{\mathfrak{f}}_0\,. \tag{5.5}$$

The Pohlmeyer reduction procedure involves solving the Virasoro constraints using a constant element

$$T \in \mathfrak{a} \subset \hat{\mathfrak{f}}_2\,, \tag{5.6}$$

where $\mathfrak{a}$ is the maximal abelian subalgebra of $\hat{\mathfrak{f}}_2$. [33] This element T induces a further $\mathbb{Z}_2$ decomposition on the algebra

$$\begin{aligned} \hat{\mathfrak{f}} = \hat{\mathfrak{f}}^{\perp} + \hat{\mathfrak{f}}^{\parallel}\,, \qquad \text{where} \qquad & \hat{\mathfrak{f}}^{\perp} = \{T, \{T, \hat{\mathfrak{f}}\}\}\,, \quad \hat{\mathfrak{f}}^{\parallel} = [T,\, [T,\, \hat{\mathfrak{f}}]]\,, \\ \Rightarrow \quad & [\mathfrak{f}^{\perp},\, T] = 0\,, \qquad\qquad \{\mathfrak{f}^{\parallel}, T\} = 0\,. \end{aligned} \tag{5.7}$$

The reduced theory is then identified with a fermionic extension of a gauged WZW model with an integrable potential (1.1). The gauged WZW model is based on the coset space G/H, where the algebra of H is given by

$$\mathfrak{h} = \hat{\mathfrak{f}}_0^{\perp}\,. \tag{5.8}$$

Let us also consider a particular sub-superalgebra of $\hat{\mathfrak{f}}$

$$\hat{\mathfrak{f}}^{\perp} = \hat{\mathfrak{h}} \ltimes T\,, \qquad \hat{\mathfrak{h}} = \mathfrak{h} \oplus \hat{\mathfrak{f}}_1^{\perp} \oplus \hat{\mathfrak{f}}_3^{\perp}\,, \tag{5.9}$$

where

$$[\hat{\mathfrak{f}}_1^{\perp},\, \hat{\mathfrak{f}}_1^{\perp}]\,,\ [\hat{\mathfrak{f}}_3^{\perp},\, \hat{\mathfrak{f}}_3^{\perp}] \subset T\,, \qquad [\hat{\mathfrak{f}}_1^{\perp},\, \hat{\mathfrak{f}}_3^{\perp}]\,,\ [\hat{\mathfrak{f}}_3^{\perp},\, \hat{\mathfrak{f}}_1^{\perp}] \subset \mathfrak{h}\,. \tag{5.10}$$

Here T behaves like a central extension as it is abelian and commutes with all other generators, (5.7). The algebra $\hat{\mathfrak{h}}$ [34] will play a rôle in the discussion of the quantum-deformed supersymmetry of the reduced theories.

5.2 Reduced $AdS_2 \times S^2$ theory

As explained in section 3.1, the reduced $AdS_2 \times S^2$ theory has manifest $\mathcal{N} = 2$ 2-d supersymmetry with the superalgebra that can be written as

$$\mathfrak{so}(1,1) \Subset ([\mathfrak{psu}(1|1)]^2 \ltimes \mathbb{R}^2)\,. \tag{5.11}$$

This agrees with the form of (5.3) with no quantum deformation. As the $\mathcal{N} = 2$ supersymmetry is manifest in the action, one should not indeed expect any quantum deformation in this case. As discussed earlier,

[32] Here U denotes the universal enveloping algebra. The subscript q then stands for the quantum deformation of this algebra, (which has no effect in the $AdS_2 \times S^2$ case). That is $U_q(\mathfrak{c})$ is the quantum group.

[33] In the reduced $AdS_n \times S^n$ theories $\mathfrak{a}$ is always 2-dimensional. In the non-degenerate case T is non-zero in both the AdS_n and S^n parts of the algebra.

[34] This algebra can be found via an Wigner-İnönü contraction, i.e. by rescaling the central extension generators by a constant and sending it to zero.

ignoring the Lorentz part of the algebra, up to central extensions all the generators of the algebra commute/anticommute. The quantum deformation of the type considered in the reduced $AdS_3 \times S^3$ theory would thus have no effect on (5.11).

The $AdS_2 \times S^2$ superstring model can be written as a GS sigma-model with the target space

$$\frac{PSU(1,1|2)}{SO(1,1) \times SO(2)} \,. \tag{5.12}$$

Here $\hat{F} = PSU(1,1|2)$ is a global symmetry and $G = SO(1,1) \times SO(2)$ is a gauge symmetry. In the group H of the Pohlmeyer reduced theory here is trivial. Then the subspaces $\hat{f}_1^{\perp}$ and $\hat{f}_3^{\perp}$ are both superalgebras equivalent to $\mathfrak{psu}(1|1)$. Therefore, here $\hat{\mathfrak{h}}$ in (5.9) is

$$\hat{\mathfrak{h}} = [\mathfrak{psu}(1|1)]^2 \,, \tag{5.13}$$

i.e. is the same algebra as $\mathfrak{c}$ in (5.3), (5.11).

5.3 Reduced $AdS_3 \times S^3$ theory

As discussed in section 4.1 and appendix B the manifest bosonic symmetry of the Lagrangian of the reduced $AdS_3 \times S^3$ theory has the following algebra

$$\mathfrak{iso}(1,1) \oplus [\mathfrak{u}(1)]^3 \oplus [\mathfrak{u}^{(g)}(1)]^3 \,, \tag{5.14}$$

where the superscript (g) denotes a gauge symmetry. The fields on which the global part of the gauge symmetry has a linear action are field redefinitions of the fields on which the global H symmetry has a linear action. Therefore, the physical symmetry acting on on-shell states is

$$\mathfrak{iso}(1,1) \oplus [\mathfrak{u}(1)]^3 \,. \tag{5.15}$$

In section 4.4 an on-shell quantum-deformed supersymmetry was shown to co-commute with the one-loop S-matrix. This quantum supersymmetry extends the physical symmetry of the theory to

$$U_q\Big(\mathfrak{so}(1,1) \inplus \big(\big[\mathfrak{u}(1) \inplus [\mathfrak{psu}(1|1)]^2\big]^2 \ltimes \mathfrak{u}(1) \ltimes \mathbb{R}^2\big)\Big) \,. \tag{5.16}$$

Due to the abelian nature of the bosonic subgroup (up to central extensions), only the action of the supersymmetry generators on two-particle states is quantum deformed.

The $AdS_3 \times S^3$ superstring theory can be written as a 2-d sigma-model with the target space [35]

$$\frac{PS[U(1,1|2) \times U(1,1|2)]}{U(1,1) \times U(2)} \,. \tag{5.17}$$

This theory has a global $\hat{F} = PS([U(1,1|2)]^2)$ symmetry and $G = U(1,1) \times U(2)$ gauge symmetry. Then in the reduced theory $H = [U(1)]^4$. As explained in appendix B one of these $U(1)$s acts trivially on all the fields and can thus be ignored leaving us with $H = [U(1)]^3$.

Once this extra $\mathfrak{u}(1)$ is projected out we find that $\hat{\mathfrak{h}}$ in (5.9) is

$$\hat{\mathfrak{h}} = \big[\mathfrak{u}(1) \inplus [\mathfrak{psu}(1|1)]^2\big]^2 \ltimes \mathfrak{u}(1) \,. \tag{5.18}$$

We see that again $\hat{\mathfrak{h}}$ is the same algebra as $\mathfrak{c}$ in (5.3),(5.16).

5.4 Reduced $AdS_5 \times S^5$ theory

Let us now try to use an analogy with the lower-dimensional cases to understand which symmetries should appear in the $AdS_5 \times S^5$ case. The $AdS_5 \times S^5$ superstring theory is based on the supercoset

$$\frac{PSU(2,2|4)}{Sp(2,2) \times Sp(4)} \,, \tag{5.19}$$

[35] Here we use a somewhat non-standard form of the supercoset leading to an equivalent Lagrangian. As explained in appendix B the symmetry analysis of this theory is more systematic if we consider the coset (5.17). For a discussion of definitions of various projections of central elements see appendix B.

i.e. here $\hat{F} = PSU(2,2|4)$ is the global symmetry and $G = Sp(2,2) \times Sp(4)$ is the gauge symmetry. The gauge group of the reduced theory is $H = [SU(2)]^4$. Then the algebra $\hat{\mathfrak{h}}$ in (5.9) is [36]

$$\hat{\mathfrak{h}} = \left[\mathfrak{psu}(2|2)\right]^2 . \tag{5.20}$$

The manifest bosonic symmetry algebra of the reduced $AdS_5 \times S^5$ theory Lagrangian is given by

$$\mathfrak{iso}(1,1) \oplus \left[\mathfrak{su}^{(g)}(2)\right]^4 , \tag{5.21}$$

where again the superscript (g) denotes gauge symmetry. There is no additional global symmetry in contrast to the abelian H case.

The perturbative S-matrix found in section 2.2.3 is constructed in such a way that it has manifest global symmetry $[SU(2)]^4$, which is the same as the global part of the gauge group. This is true at tree-level [1] and also at the one-loop level as will be shown in sections 6.1 and 6.1.1. A manifest (i.e. acting with the standard coproduct) non-abelian global symmetry of a relativistic, trigonometric S-matrix for the theories (1.1) is already in conflict [2] with the Yang-Baxter equation at the tree level.

Motivated by the $AdS_3 \times S^3$ example, where the symmetry group $\hat{\mathfrak{h}}$ was quantum-deformed, one may conjecture that the same should happen also in the $AdS_5 \times S^5$ case, i.e. the S-matrix should be invariant under the corresponding quantum group

$$U_q\Big(\mathfrak{so}(1,1) \in \left(\left[\mathfrak{psu}(2|2)\right]^2 \ltimes \mathbb{R}^2\right)\Big) , \tag{5.22}$$

where we have replaced $\mathfrak{c}$ in (5.3) with $\hat{\mathfrak{h}}$ from (5.20).

An R-matrix invariant under $U_q(\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3)$ has been studied in [36, 37]. It was observed that there is a particular classical limit of the R-matrix [37] that bears strong resemblance to the tree-level S-matrix found in [1]. We will extend this limit to all orders in $1/k$ in section 6.2, finding a relativistic trigonometric R-matrix satisfying unitarity, crossing and the Yang-Baxter equation. In this limit the third central extension vanishes. The resulting R-matrix has similarities to the one-loop S-matrix of section 2.2.3. It is then natural to consider this R-matrix as a candidate for the physical S-matrix of the perturbative excitations of the reduced $AdS_5 \times S^5$ theory.

Unlike the reduced $AdS_3 \times S^3$ theory here the group $H = [SU(2)]^4$ is non-abelian and therefore the quantum deformation will non-trivially affect its action on the two-particle states (the action of H on the one-loop perturbative S-matrix was assumed to be given by the standard coproduct). Understanding the origin of this quantum deformation is an important open question. More generally, this question applies also to similar bosonic models with a non-abelian gauge group [2, 14, 15, 17] discussed in appendix G.

6 S-matrix of the reduced $AdS_5 \times S^5$ theory

In this section we finally consider the case of prime interest – the reduced $AdS_5 \times S^5$ theory – with the aim of understanding the structure of the corresponding quantum S-matrix.

We shall first demonstrate that the group factorisation property of the perturbative S-matrix (see section 2.2.3), first observed at tree-level in [1], continues to one-loop level. The factorised S-matrix can then be compared with the quantum-deformed $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ R-matrix of [36]. A particular classical limit of this R-matrix was identified in [37] whose form is very similar to that of the tree-level S-matrix in [1]. We will extend this limit to all orders and show, in particular, that this similarity continues also to the one-loop order. It is then natural to consider this R-matrix as a candidate for the physical S-matrix of the perturbative excitations of the reduced $AdS_5 \times S^5$ theory.

Other than the similarity with the tree-level S-matrix the main motivation for considering this R-matrix is an analogy with the S-matrix in the $AdS_3 \times S^3$ case which is invariant under a quantum-deformed supersymmetry. The choice of the symmetry algebra, $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^2$, is explained in section 5. Also, in bosonic theories similar to (1.1) with a non-abelian group H the quantum deformation has been conjectured to be the physical symmetry of the theory [14, 15, 17].

[36] This same sub-superalgebra arises also when considering the expansion of the superstring action around the BMN vacuum: the manifest symmetry of the Lagrangian is broken to $[SU(2)]^4$ while the on-shell symmetry (under which the superstring S-matrix is invariant) is precisely $\hat{\mathfrak{h}}$.

The one-loop S-matrix computed in [1] and in section 2.2.3 is invariant under the maximal bosonic subalgebra of $[\mathfrak{psu}(2|2)]^2$. This subalgebra is non-abelian. However, an S-matrix invariant under the quantum deformation is not invariant under this non-abelian symmetry with the standard coproduct for the bosonic generators. Hence there are differences between the perturbative S-matrix originating from the action (1.1) and the quantum-deformed S-matrix seen already at the tree level [37, 1]. This is not surprising as the quantum-deformed S-matrix satisfies the Yang-Baxter equation while the perturbative H-invariant one-loop S-matrix does not. We conclude this section by investigating a relation between these two S-matrices.

6.1 Perturbative S-matrix at one-loop order

In the reduced $AdS_5 \times S^5$ theory [3] we have $G = Sp(2,2) \times Sp(4)$ and the gauge group $H = [SU(2)]^4$. The Lagrangian (2.3) is written with a manifest global $SO(4)$ symmetry. This symmetry is a subgroup of the global part of the gauge group H. As the $A_+ = 0$ gauge fixing preserves the global part of H the Lagrangian is expected to be invariant under the full $H = [SU(2)]^4$ global symmetry. This symmetry can be made manifest by using the field redefinitions [1] [37]

$$\begin{aligned} Y_m &= (\bar\sigma_m)^{\dot a a} Y_{a\dot a} \, , & Y_{a\dot a} &= (\sigma_m)_{a\dot a} Y_m \, , \\ Z_m &= (\bar\sigma_m)^{\dot\alpha\alpha} Z_{\alpha\dot\alpha} \, , & Z_{\alpha\dot\alpha} &= (\sigma_m)_{\alpha\dot\alpha} Z_m \, , \\ \zeta_m &= (\bar\sigma_m)^{\dot\alpha a} \zeta_{a\dot\alpha} \, , & Y_{a\dot\alpha} &= (\sigma_m)_{a\dot\alpha} Y_m \, , \\ \chi_m &= (\bar\sigma_m)^{\dot a\alpha} \chi_{\alpha\dot a} \, , & \chi_{\alpha\dot a} &= (\sigma_m)_{\alpha\dot a} \chi_m \, , \end{aligned} \tag{6.1}$$

where $(Y_{a\dot a})^* = Y^{\dot a a}$, etc. The translation of (2.3) into the manifestly $[SU(2)]^4$ invariant form is [1]

$$\begin{aligned} \mathcal{L}_5 \;=\; & \frac{1}{2}\partial_+ Y_{a\dot a}\partial_- Y^{\dot a a} - \frac{\mu^2}{2} Y_{a\dot a}Y^{\dot a a} + \frac{1}{2}\partial_+ Z_{\alpha\dot\alpha}\partial_- Z^{\dot\alpha\alpha} - \frac{\mu^2}{2} Z_{\alpha\dot\alpha}Z^{\dot\alpha\alpha} \\ & + \frac{i}{2}\zeta_{L\,a\dot\alpha}\partial_+ \zeta_L{}^{\dot\alpha a} + \frac{i}{2}\zeta_{R\,a\dot\alpha}\partial_- \zeta_R{}^{\dot\alpha a} - i\mu \zeta_{L\,a\dot\alpha}\zeta_R{}^{\dot\alpha a} \\ & + \frac{i}{2}\chi_{L\,\alpha\dot a}\partial_+ \chi_L{}^{\dot a\alpha} + \frac{i}{2}\chi_{R\,\alpha\dot a}\partial_- \chi_R{}^{\dot a\alpha} - i\mu \chi_{L\,\alpha\dot a}\chi_R{}^{\dot a\alpha} \\ & + \frac{\pi}{2k}\Big[-\frac{2}{3}\big(Y_{a\dot a}Y^{\dot a a}\partial_+ Y_{b\dot b}\partial_- Y^{\dot b b} - Y_{a\dot a}\partial_+ Y^{\dot a a} Y_{b\dot b}\partial_- Y^{\dot b b} + \frac{\mu^2}{2} Y_{a\dot a}Y^{\dot a a}Y_{b\dot b}Y^{\dot b b}\big) \\ & \qquad + \frac{2}{3}\big(Z_{\alpha\dot\alpha}Z^{\dot\alpha\alpha}\partial_+ Z_{\beta\dot\beta}\partial_- Z^{\dot\beta\beta} - Z_{\alpha\dot\alpha}\partial_+ Z^{\dot\alpha\alpha} Z_{\beta\dot\beta}\partial_- Z^{\dot\beta\beta} + \frac{\mu^2}{2} Z_{\alpha\dot\alpha}Z^{\dot\alpha\alpha}Z_{\beta\dot\beta}Z^{\dot\beta\beta}\big) \\ & \qquad + i\big(\zeta_{L\,a\dot\alpha}\zeta_L{}^{\dot\alpha b}Y^{\dot b a}\partial_+ Y_{b\dot b} + \zeta_{R\,a\dot\alpha}\zeta_R{}^{\dot\alpha b}Y^{\dot b a}\partial_- Y_{b\dot b} + \mu\,\zeta_{R\,a\dot\alpha}\zeta_L{}^{\dot\alpha a}Y_{b\dot b}Y^{\dot b b}\big) \\ & \qquad - i\big(\zeta_{L\,a\dot\alpha}\zeta_L{}^{\dot\beta a}Z^{\dot\alpha\beta}\partial_+ Z_{\beta\dot\beta} + \zeta_{R\,a\dot\alpha}\zeta_R{}^{\dot\beta a}Z^{\dot\alpha\beta}\partial_- Z_{\beta\dot\beta} + \mu\,\zeta_{R\,a\dot\alpha}\zeta_L{}^{\dot\alpha a}Z_{\beta\dot\beta}Z^{\dot\beta\beta}\big) \\ & \qquad + i\big(\chi_{L\,\alpha\dot a}\chi_L{}^{\dot b\alpha}Y^{\dot a b}\partial_+ Y_{b\dot b} + \chi_{R\,\alpha\dot a}\chi_R{}^{\dot b\alpha}Y^{\dot a b}\partial_- Y_{b\dot b} + \mu\,\chi_{R\,\alpha\dot a}\chi_L{}^{\dot a\alpha}Y_{b\dot b}Y^{\dot b b}\big) \\ & \qquad - i\big(\chi_{L\,\alpha\dot a}\chi_L{}^{\dot a\beta}Z^{\dot\beta\alpha}\partial_+ Z_{\beta\dot\beta} + \chi_{R\,\alpha\dot a}\chi_R{}^{\dot a\beta}Z^{\dot\beta\alpha}\partial_- Z_{\beta\dot\beta} + \mu\,\chi_{R\,\alpha\dot a}\chi_L{}^{\dot a\alpha}Z_{\beta\dot\beta}Z^{\dot\beta\beta}\big) \\ & \qquad + 4i\mu\big(\zeta_{R\,a\dot\alpha}\chi_{L\,\beta\dot b}Y^{\dot b a}Z^{\dot\alpha\beta} - \chi_{R\,\alpha\dot a}\zeta_{L\,b\dot\beta}Y^{\dot a b}Z^{\dot\beta\alpha}\big) \\ & \qquad + 2\big(\zeta_{L\,a\dot\alpha}\zeta_{L\,b\dot\beta}\zeta_R{}^{\dot\alpha b}\zeta_R{}^{\dot\beta a} - \chi_{L\,\alpha\dot a}\chi_{L\,\beta\dot\beta}\chi_R{}^{\dot a\beta}\chi_R{}^{\dot b\alpha}\big)\Big] + \mathcal{O}(k^{-2}) \, . \end{aligned} \tag{6.2}$$

6.1.1 Group factorisation of the S-matrix

Consider the following factorised S-matrix [1, 34]

$$\mathbb{S}\,|\Phi_{A\dot A}(p_1)\Phi_{B\dot B}(p_2)\rangle = (-1)^{[\dot A][B]+[\dot C][D]}\; S_{AB}^{CD}\; S_{\dot A\dot B}^{\dot C\dot D}\; |\Phi_{C\dot C}(p_1)\Phi_{D\dot D}(p_2)\rangle \;, \tag{6.3}$$

[37] The 2-indices are raised and lowered with the antisymmetric tensors ϵ^{ab}, etc., i.e. $F^a = \epsilon^{ab}F_b$, $F_b = \epsilon_{bc}F^c$. Dotted and undotted indices are assumed to be completely independent. We use the convention that $\epsilon^{12} = 1$, $\epsilon_{12} = -1$, $\epsilon^{ab}\epsilon_{bc} = \delta^a_c$ and the rescaled set of Pauli matrices

$$\sigma^1 = \bar\sigma^1 = \frac{1}{\sqrt2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \;, \quad \sigma^2 = -\bar\sigma^2 = \frac{1}{\sqrt2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \;, \quad \sigma^3 = -\bar\sigma^3 = \frac{1}{\sqrt2}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix} \;, \quad \sigma^4 = -\bar\sigma^4 = \frac{1}{\sqrt2}\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix} .$$

$$
S_{AB}^{CD} = \begin{cases} K_1\delta_a^c\delta_b^d + K_2\delta_a^d\delta_b^c\,, \\ K_3\delta_\alpha^\gamma\delta_\beta^\delta + K_4\delta_\alpha^\delta\delta_\beta^\gamma\,, \\ K_5\epsilon_{ab}\epsilon^{\gamma\delta}\,, \quad K_6\epsilon_{\alpha\beta}\epsilon^{cd}\,, \\ K_7\delta_a^d\delta_\beta^\gamma\,, \quad\;\; K_8\delta_\alpha^\delta\delta_b^c\,, \\ K_9\delta_a^c\delta_\beta^\delta\,, \quad\;\; K_{10}\delta_\alpha^\gamma\delta_b^d\,, \end{cases} \tag{6.4}
$$

where K_i are functions of θ and the coupling k. Here $\Phi_{A\dot{A}}$ (with $A=(a,\alpha)$ and $\dot{A}=(\dot{a},\dot{\alpha})$, where $a,\alpha,\dot{a},\dot{\alpha}$ are indices of fundamental representations of the four $SU(2)$ groups comprising the global $[SU(2)]^4$ symmetry) encodes the fields $Y_{a\dot{a}}\,,\ Z_{\alpha\dot{\alpha}}\,,\ \zeta_{a\dot{\alpha}}\,,\ \chi_{\alpha\dot{a}}$. As for the reduced $AdS_3\times S^3$ theory (4.15) we assume the fermionic grading

$$
[a] = [\dot{a}] = 0\,, \qquad [\alpha] = [\dot{\alpha}] = 1\,. \tag{6.5}
$$

As in the reduced $AdS_3\times S^3$ theory it is useful to consider "half" of this factorised S-matrix (6.4) as acting on a single field

$$
\mathbb{S}\,|\Phi_A(p_1)\Phi_B(p_2)\rangle = S_{AB}^{CD}(\theta,k)\,|\Phi_C(p_1)\Phi_D(p_2)\rangle\ , \qquad \Phi_A = (\phi_a,\psi_\alpha)\,, \tag{6.6}
$$

where ϕ_a are bosonic and ψ_α are fermionic. This S-matrix should satisfy the usual physical requirements of unitarity and crossing. The unitarity and crossing relations are the same as in (4.22), (4.23) and (4.24). The crossed S-matrix $\bar{S}_{AB}^{CD}(\theta,k)$ is identical to $S_{AB}^{CD}(\theta,k)$ in (6.6) except with $(K_5,\,K_6,\,K_7,\,K_8)$ replaced by $i(K_5,\,K_6,\,K_7,\,K_8)$. Crossing symmetry requires the following relations between the functions K_i,

$$
\begin{aligned}
&K_1(i\pi-\theta,k) = K_1(\theta,k)+K_2(\theta,k)\,, \qquad K_2(i\pi-\theta,k) = -K_2(\theta,k)\,, \\
&K_5(i\pi-\theta,k) = iK_7(\theta,k)\,, \qquad\qquad\quad\; K_7(i\pi-\theta,k) = -iK_5(\theta,k)\,, \\
&\qquad\qquad K_9(i\pi-\theta,k) = K_9(\theta,k)\,,
\end{aligned} \tag{6.7}
$$

and similarly for $K_3,\,K_4,\,K_6,\,K_8,\,K_{10}$.

For consistency (for example between $\mathbb{S}\,|Y_m(p_1)\zeta_n(p_2)\rangle$ and $\mathbb{S}\,|\zeta_m(p_1)Y_n(p_2)\rangle$, see appendix E) the functions K_i should also obey the following conjugation relations

$$
\begin{aligned}
K_1(\theta,k) &= K_1^*(-\theta,k)\,, \qquad & K_3(\theta,k) &= K_3^*(-\theta,k)\,, \\
K_2(\theta,k) &= K_2^*(-\theta,k)\,, \qquad & K_4(\theta,k) &= K_4^*(-\theta,k)\,, \\
K_5(\theta,k) &= -K_5^*(-\theta,k)\,, \qquad & K_6(\theta,k) &= -K_6^*(-\theta,k)\,, \\
K_7(\theta,k) &= K_8^*(-\theta,k)\,, \qquad & K_9(\theta,k) &= K_{10}^*(-\theta,k)\,.
\end{aligned} \tag{6.8}
$$

The Lagrangian (2.3) has also the $\mathbb{Z}_2$ symmetry (4.27) implying the following relations

$$
\begin{aligned}
&K_1(\theta,k) = K_3(\theta,-k)\,, \qquad\quad K_2(\theta,k) = K_4(\theta,-k)\,, \\
&K_5(\theta,k) = -K_6(\theta,-k)\,, \qquad K_7(\theta,k) = -K_8(\theta,-k)\,, \\
&\qquad\qquad K_9(\theta,k) = K_{10}(\theta,-k)\,.
\end{aligned} \tag{6.9}
$$

In appendix E equation (6.3) is expanded with arbitrary K_i and rewritten in the original $SO(4)$ notation to enable comparison to the S-matrix of section 2.2.3. Along with other relations between the parametrising functions, the relation between $YYZZ$ and $\zeta\zeta\chi\chi$ amplitudes (f_3 and f_5 functions) means the one-loop S-matrix of section 2.2.3 factorises as in (6.3), (6.4) with K_i given by

$$
K_i = \ p_0(\theta,k\,;\tfrac{1}{2})\ \hat{K}_i\ , \tag{6.10}
$$

$$
\begin{aligned}
\hat{K}_1(\theta,k) =& \hat{K}_3(\theta,-k) = 1 + \frac{i\pi}{2k}\tanh\frac{\theta}{2} - \frac{5\pi^2}{8k^2} - \frac{i\pi\theta}{2k^2} + \mathcal{O}(k^{-3}) \\
\hat{K}_2(\theta,k) =& \hat{K}_4(\theta,-k) = -\frac{i\pi}{k}\coth\theta + \frac{\pi^2}{2k^2} + \frac{i\pi\theta}{k^2} + \mathcal{O}(k^{-3}) \\
\hat{K}_5(\theta,k) =& -\hat{K}_6(\theta,-k) = -\frac{i\pi}{2k}\,\mathrm{sech}\,\frac{\theta}{2} + \mathcal{O}(k^{-3}) \\
\hat{K}_7(\theta,k) =& -\hat{K}_8(\theta,-k) = -\frac{i\pi}{2k}\,\mathrm{cosech}\,\frac{\theta}{2} + \mathcal{O}(k^{-3}) \\
\hat{K}_9(\theta,k) =& \hat{K}_{10}(\theta,-k) = 1 + \mathcal{O}(k^{-3})\ .
\end{aligned} \tag{6.11}
$$

Here the phase factor $p_0(\theta, k\,;\frac{1}{2})$ was defined in (2.15) and represents the square root of the factor in (2.20) (where we considered the full S-matrix rather than "half" of the factorised S-matrix).

The choice of the phase factor (2.20) ensures that $\hat{K}_{9,10} = 1 + \mathcal{O}(k^{-3})$ to one-loop order. This choice is convenient for comparing to the quantum-deformed S-matrix. From the expanded S-matrix in appendix E we see that like in the reduced $AdS_2 \times S^2$ case (but unlike the reduced $AdS_3 \times S^3$ case) a phase factor can be extracted such that both the amplitudes of $Y_m Z_n \to Y_m Z_n$ and $\zeta_m \chi_n \to \zeta_m \chi_n$ scattering processes and $K_{9,10}$ are all equal to 1.

The $AdS_2 \times S^2$ and $AdS_5 \times S^5$ superstring sigma models are of the same type being based on the supergroup

$$PSU(n,n|2n)\,, \qquad n = 1,\ 2\ . \tag{6.12}$$

It is thus natural to expect that the S-matrices of the corresponding reduced theories follow the same pattern, in particular, their phase factors are similar. The phase factors that we extracted in these cases in sections 2.2.1 and 2.2.3 (given by the amplitude for the $Y_m Z_n \to Y_m Z_n$ and $\zeta_m \chi_n \to \zeta_m \chi_n$ processes) were equal. We thus conjecture that the phase factor of the reduced $AdS_5 \times S^5$ theory should be given by the phase factor (3.27) of the reduced $AdS_2 \times S^2$ theory, again with $\Delta = \frac{\pi}{k}$ as in (3.25). Further justification for this choice is presented in section 6.2, where we find that this is precisely the phase factor that follows from solving the conditions of unitarity and crossing for the quantum-deformed S-matrix.

The one-loop perturbative S-matrix (6.4), (6.6), (6.11) satisfies the expected unitarity, crossing and conjugation relations, (4.22), (4.23), (6.7), (6.8) with the crossed S-matrix given above (6.7). Substituting the one-loop S-matrix (6.4) and (6.11) into the YBE, which has the same form as (3.15), one finds that the result is non-zero. This happened also for the bosonic models with a non-abelian symmetry H [2] where the rôle of the Yang-Baxter equation requires further study. In the next subsection we will consider the closely related quantum-deformed S-matrix [36, 37] which by construction satisfies the Yang-Baxter equation (3.15).

Let us make two additional comments. In the purely bosonic theories, the coupling k (level of WZW theory) is generally shifted by a constant in certain exact quantum relations. This shift is absent though in 2-d supersymmetric models. There appears to be no shift of k also in the reduced $AdS_2 \times S^2$ and $AdS_3 \times S^3$ theories discussed in sections 3 and 4. The same should be true also in the reduced $AdS_5 \times S^5$ theory.

In the reduced $AdS_3 \times S^3$ theory a quantum counterterm was required to restore integrability, i.e. the satisfaction of the YBE at one-loop order, see section 4.2. In the reduced $AdS_5 \times S^5$ theory counterterms are not required to restore the group factorisation of the one-loop S-matrix [38] and the similarity with the quantum-deformed S-matrix of section 6.2 suggests that indeed no additional local counterterms should be present here. [39]

6.2 Quantum-deformed $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ symmetric S-matrix

In [36] the fundamental R-matrix associated to the quantum deformation of the universal enveloping algebra (U_q) of the centrally extended superalgebra $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ was constructed. In appendix F we generalise to all orders the trigonometric relativistic classical limit identified in [37] that exhibited similarities to the tree-level S-matrix of the reduced $AdS_5 \times S^5$ theory [1]. The trigonometric relativistic limit corresponds to (i) taking the global symmetry parameter g of [36] to infinity and (ii) identifying the quantum deformation parameter q with the coupling k as in (4.47) [40]

$$q = \exp\Big(-\frac{i\pi}{k}\Big)\,. \tag{6.13}$$

In this limit one of the central extensions vanishes leaving us with the symmetry group (5.22) expected from the arguments of section 5 .

The quantum-deformed S-matrix in this limit takes the form

$$\mathcal{S}\,|\phi_1\phi_1\rangle = \big(J_1 + J_2\big)\,|\phi_1\phi_1\rangle$$

[38] This is not the case for the bosonic $G/H = SO(5)/SO(4)$ theory discussed in [5]. There the quantum counterterms were required to restore group factorisation under $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ at one-loop order.

[39] In appendix C a particular functional determinant based on the group structure of the reduced theories (section 5.1) is proposed. The contribution of this functional determinant to the one-loop S-matrix vanishes in the reduced $AdS_2 \times S^2$ and $AdS_5 \times S^5$ theories while giving the required correction (4.9) in the reduced $AdS_3 \times S^3$ theory.

[40] Such a parametrisation of the quantum deformation parameter is familiar from quantum group structures in theories based on (deformations of) WZW model (see, e.g., [52, 14, 15]).

$$
\begin{aligned}
\mathcal{S}\,|\phi_1\phi_2\rangle &= J_1 \sec\frac{\pi}{k}\,|\phi_1\phi_2\rangle + \big(J_2 - iJ_1\tan\frac{\pi}{k}\big)\,|\phi_2\phi_1\rangle - J_5\sec\frac{\pi}{k}\,|\psi_3\psi_4\rangle + J_5(1+i\tan\frac{\pi}{k})\,|\psi_4\psi_3\rangle \\
\mathcal{S}\,|\phi_2\phi_1\rangle &= J_1 \sec\frac{\pi}{k}\,|\phi_2\phi_1\rangle + \big(J_2 + iJ_1\tan\frac{\pi}{k}\big)\,|\phi_1\phi_2\rangle - J_5\sec\frac{\pi}{k}\,|\psi_4\psi_3\rangle + J_5(1-i\tan\frac{\pi}{k})\,|\psi_3\psi_4\rangle \\
\mathcal{S}\,|\phi_2\phi_2\rangle &= \big(J_1+J_2\big)\,|\phi_2\phi_2\rangle \\
\mathcal{S}\,|\psi_3\psi_3\rangle &= \big(J_3+J_4\big)\,|\psi_3\psi_3\rangle \\
\mathcal{S}\,|\psi_3\psi_4\rangle &= J_3 \sec\frac{\pi}{k}\,|\psi_3\psi_4\rangle + \big(J_4 - iJ_3\tan\frac{\pi}{k}\big)\,|\psi_4\psi_3\rangle - J_6\sec\frac{\pi}{k}\,|\phi_1\phi_2\rangle + J_6(1+i\tan\frac{\pi}{k})\,|\phi_2\phi_1\rangle \\
\mathcal{S}\,|\psi_4\psi_3\rangle &= J_3 \sec\frac{\pi}{k}\,|\psi_4\psi_3\rangle + \big(J_4 + iJ_3\tan\frac{\pi}{k}\big)\,|\psi_3\psi_4\rangle - J_6\sec\frac{\pi}{k}\,|\phi_2\phi_1\rangle + J_6(1-i\tan\frac{\pi}{k})\,|\phi_1\phi_2\rangle \\
\mathcal{S}\,|\psi_4\psi_4\rangle &= \big(J_3+J_4\big)\,|\psi_4\psi_4\rangle \\
\mathcal{S}\,|\phi_a\psi_\beta\rangle &= J_7\;\delta_a^d\delta_\beta^\gamma\,|\psi_\gamma\phi_d\rangle + J_9\;\delta_a^c\delta_\beta^\delta\,|\phi_c\psi_\delta\rangle \\
\mathcal{S}\,|\psi_\alpha\phi_b\rangle &= J_8\;\delta_\alpha^\delta\delta_b^c\,|\phi_c\psi_\delta\rangle + J_{10}\;\delta_\alpha^\gamma\delta_b^d\,|\psi_\gamma\phi_d\rangle\ ,
\end{aligned}
\tag{6.14}
$$

where

$$
\begin{aligned}
J_{1,3}(\theta,k) &= P_0(\theta,k)\cos\frac{\pi}{k}\,\mathrm{sech}\,\frac{\theta}{2}\cosh\Big(\frac{\theta}{2}\pm\frac{i\pi}{2k}\Big) \\
J_{2,4}(\theta,k) &= \mp iP_0(\theta,k)\Big[1-\cos\frac{\pi}{k}+\cosh\theta+\cosh\Big(\theta\pm\frac{i\pi}{k}\Big)\Big]\sin\frac{\pi}{2k}\,\mathrm{cosech}\,\theta \\
J_{5,6}(\theta,k) &= -iP_0(\theta,k)\cos\frac{\pi}{k}\sin\frac{\pi}{2k}\,\mathrm{sech}\,\frac{\theta}{2} \\
J_{7,8}(\theta,k) &= -iP_0(\theta,k)\sin\frac{\pi}{2k}\,\mathrm{cosech}\,\frac{\theta}{2} \\
J_{9,10}(\theta,k) &= P_0(\theta,k)
\end{aligned}
\tag{6.15}
$$

The functions J_i do not parametrise the quantum-deformed S-matrix in the same way as the functions K_i in (6.11) parametrise the perturbative S-matrix given in (6.4): there is an additional dependence on $q=\exp(-\frac{i\pi}{k})$. Consequently, the $[SU(2)]^2$ global symmetry is broken to $[U(1)]^2$. The quantum-deformed S-matrix satisfies the Yang-Baxter equation.

Extracting the phase factor $P_0(\theta,k)$,

$$
J_i(\theta,k) = P_0(\theta,k)\ \hat{J}_i(\theta,k)\ , \tag{6.16}
$$

the $1/k$ expansion of the functions $\hat{J}_i(\theta,k)$ is given, to "one-loop" order, by

$$
\begin{aligned}
\hat{J}_1(\theta,k) =& \hat{J}_3(\theta,-k) = 1+\frac{i\pi}{2k}\tanh\frac{\theta}{2}-\frac{5\pi^2}{8k^2}+\mathcal{O}(k^{-3}) \\
\hat{J}_2(\theta,k) =& \hat{J}_4(\theta,-k) = -\frac{i\pi}{k}\coth\theta+\frac{\pi^2}{4k^2}+\mathcal{O}(k^{-3}) \\
\hat{J}_5(\theta,k) =& -\hat{J}_6(\theta,-k) = -\frac{i\pi}{2k}\,\mathrm{sech}\,\frac{\theta}{2}+\mathcal{O}(k^{-3}) \\
\hat{J}_7(\theta,k) =& -\hat{J}_8(\theta,-k) = -\frac{i\pi}{2k}\,\mathrm{cosech}\,\frac{\theta}{2}+\mathcal{O}(k^{-3}) \\
\hat{J}_9(\theta,k) =& \hat{J}_{10}(\theta,-k) = 1+\mathcal{O}(k^{-3})\ .
\end{aligned}
\tag{6.17}
$$

There is a strong similarity with (6.11). However, $K_{1,2,3,4}$ contain some extra θ-dependent terms. The functions J_i in (6.15) do not satisfy the classical crossing symmetry relations obeyed by K_i (6.7), rather they satisfy a set quantum-deformed relations (6.25) given below.

6.2.1 Phase factor

To facilitate comparison with the one-loop S-matrix of sections 2.2.3 and 6.1.1, the phase factor P_0 has been chosen such that $\hat{J}_9=\hat{J}_{10}=1$. This phase factor is fixed by the physical requirements of unitarity, crossing and matrix unitarity [36] (see appendix F). We give these relations in terms of the tensor function $S_{q\,AB}^{\;CD}(\theta,k)$ defined by

$$
\mathcal{S}\,|\Phi_A(p_1)\Phi_B(p_2)\rangle = S_{q\,AB}^{\;CD}(\theta,k)\,|\Phi_C(p_1)\Phi_D(p_2)\rangle\ . \tag{6.18}
$$

Unitarity implies (see (F.12))

$$(-1)^{[C][D]+[E][F]} S_{q\,AB}^{\;CD}(\theta,k)\ S_{q\,DC}^{\;FE}(-\theta,k) = \delta_A^E \delta_B^F\,. \tag{6.19}$$

Substituting the quantum-deformed S-matrix (6.14), (6.15) into (6.19) gives

$$P_0(\theta,k)\ P_0(-\theta,k) = \frac{\sinh^2\frac{\theta}{2}}{\sinh^2\frac{\theta}{2}+\sin^2\frac{\pi}{2k}}\,. \tag{6.20}$$

To formulate the crossing constraint let us introduce the crossed quantum-deformed S-matrix $\bar{S}_{q\,AB}^{\;CD}(\theta,k)$ which is identical to $S_{q\,AB}^{\;CD}(\theta,k)$ except with $(J_5,\,J_6,\,J_7,\,J_8)$ replaced by $i(J_5,\,J_6,\,J_7,\,J_8)$. Then the crossing symmetry condition (F.15) implies

$$\sum_{C,D,G,H=1}^{4} S_{q\,AB}^{\;CD}(\theta,k)\ C_C^G\ \bar{S}_{q\,HD}^{\;GF}(i\pi+\theta,k)\ C^{-1}{}_H^E = \delta_A^E\delta_B^F\,. \tag{6.21}$$

In view of the unitarity relation (6.19) this can be rewritten in the usual form

$$S_{q\,AB}^{\;CD}(\theta,k) = \sum_{E,F=1}^{4} (-1)^{[A][B]+[C][D]}\ C_B^E\ \bar{S}_{q\,FA}^{\;EC}(i\pi-\theta,k)\ C^{-1}{}_F^D\,. \tag{6.22}$$

The charge conjugation matrix C_A^B is defined by

$$\begin{gathered}\mathcal{C}\,|\Phi_A\rangle = C_A^B\,|\Phi_B\rangle\ ,\\ \mathcal{C}\,|\phi_1\rangle = -q^{\frac12}\,|\phi_2\rangle\ ,\qquad \mathcal{C}\,|\psi_3\rangle = -q^{\frac12}\,|\psi_4\rangle\ ,\qquad \mathcal{C}\,|\phi_2\rangle = q^{-\frac12}\,|\phi_1\rangle\ ,\qquad \mathcal{C}\,|\psi_4\rangle = q^{-\frac12}\,|\psi_3\rangle\ ,\end{gathered} \tag{6.23}$$

and similarly for $C^{-1}{}_A^B$. Substituting the quantum-deformed S-matrix (6.14), (6.15) into (6.19) gives the crossing relation for the phase factor

$$P_0(i\pi-\theta,k) = P_0(\theta,k)\,. \tag{6.24}$$

The crossing symmetry also requires the following relations between the functions J_i,

$$\begin{gathered}J_1(i\pi-\theta,k) = \cos\frac{\pi}{k}\big[J_1(\theta,k)+J_2(\theta,k)\big]\,,\qquad J_2(i\pi-\theta,k) = -\cos\frac{\pi}{k}\big[J_2(\theta,k)-\tan^2\frac{\pi}{k}J_1(\theta,k)\big]\\ J_5(i\pi-\theta,k) = i\cos\frac{\pi}{k}J_7(\theta,k)\,,\qquad J_7(i\pi-\theta,k) = -i\sec\frac{\pi}{k}J_5(\theta,k)\\ J_9(i\pi-\theta,k) = J_9(\theta,k)\,,\end{gathered} \tag{6.25}$$

and similarly for $J_3,\,J_4,\,J_6,\,J_8,\,J_{10}$. The conjugation relations

$$\begin{aligned} J_1(\theta,k) &= J_1^*(-\theta,k)\,, & J_3(\theta,k) &= J_3^*(-\theta,k)\,,\\ J_2(\theta,k) &= J_2^*(-\theta,k)\,, & J_4(\theta,k) &= J_4^*(-\theta,k)\,,\\ J_5(\theta,k) &= -J_5^*(-\theta,k)\,, & J_6(\theta,k) &= -J_6^*(-\theta,k)\,,\\ J_7(\theta,k) &= J_8^*(-\theta,k)\,, & J_9(\theta,k) &= J_{10}^*(-\theta,k)\,. \end{aligned} \tag{6.26}$$

still hold as they did for the functions K_i (6.8) as long as the phase factor satisfies

$$P_0(\theta,k) = P_0^*(-\theta,k)\,. \tag{6.27}$$

To summarize, the trigonometric relativistic quantum-deformed S-matrix (6.14), (6.15) is consistent with unitarity and the quantum-deformed crossing symmetry provided the phase factor $P_0(\theta,k)$ satisfies the following constraints

$$\begin{gathered}P_0(\theta,k)P_0(-\theta,k) = \frac{\sinh^2\frac{\theta}{2}}{\sinh^2\frac{\theta}{2}+\sin^2\frac{\pi}{2k}}\,,\\ P_0(i\pi-\theta,k) = P_0(\theta,k)\,,\qquad P_0(\theta,k) = P_0^*(-\theta,k)\,.\end{gathered} \tag{6.28}$$

In section 6.1.1 $P(\theta,\frac{\pi}{k})$ in (3.27) was conjectured to be a candidate for the phase factor of the reduced $AdS_5\times S^5$ theory based on the one-loop computation and group-theory arguments. Here we are considering the factorised S-matrix so that the corresponding phase factor P_0 is the square root of P, i.e.

$$P_0(\theta,k)=\sqrt{P(\theta,\frac{\pi}{k})}=\sqrt{\frac{\sinh\theta+i\sin\frac{\pi}{k}}{\sinh\theta-i\sin\frac{\pi}{k}}}\;R(\theta,\frac{\pi}{k})\,,\qquad R(\theta,\Delta)=\frac{\sinh\theta-i\sin\Delta}{\sinh\theta+i\sin\Delta}\;Y(\theta,\Delta)\;Y(i\pi-\theta,\Delta)\,,$$
$$Y(\theta,\Delta)=\prod_{l=1}^{\infty}\frac{\Gamma\big(\frac{\Delta}{2\pi}-\frac{i\theta}{2\pi}+l\big)\Gamma\big(-\frac{\Delta}{2\pi}-\frac{i\theta}{2\pi}+l-1\big)\Gamma\big(-\frac{i\theta}{2\pi}+l-\frac{1}{2}\big)\Gamma\big(-\frac{i\theta}{2\pi}+l+\frac{1}{2}\big)}{\Gamma\big(\frac{\Delta}{2\pi}-\frac{i\theta}{2\pi}+l+\frac{1}{2}\big)\Gamma\big(-\frac{\Delta}{2\pi}-\frac{i\theta}{2\pi}+l-\frac{1}{2}\big)\Gamma\big(-\frac{i\theta}{2\pi}+l-1\big)\Gamma\big(-\frac{i\theta}{2\pi}+l\big)}\,,\quad \Delta(k)=\frac{\pi}{k}\,. \tag{6.29}$$

From the relations (3.31) we see that this phase factor satisfies the unitarity, crossing and conjugation relations (6.28).

The small θ expansion of the phase factor (6.29) is

$$P_0(\theta,k)=-\frac{\theta}{2\sin\frac{\pi}{2k}}\mathrm{sign}(\sin\frac{\pi}{k})+\mathcal{O}(\theta^2)\,, \tag{6.30}$$

implying the initial condition for the quantum-deformed S-matrix (for $k>0$)

$$S_{q\,AB}^{\ CD}(0,k)=i(-1)^{[C][D]}\delta_A^D\delta_B^C\ \mathrm{sign}(\sin\frac{\pi}{k})\,. \tag{6.31}$$

The quantum-deformed S-matrix (6.14), (6.15) satisfies the graded Yang-Baxter equation (3.15) by construction [36]. We have also checked this explicitly.

The quantum-deformed S-matrix (6.14), (6.15) along with the phase factor (6.29) is thus a candidate for the physical S-matrix for the perturbative excitations of the reduced $AdS_5\times S^5$ theory. There are many additional properties of this S-matrix to investigate, for example, the pole structure.

6.2.2 Quantum-deformed symmetry

Let us review the action of the quantum-deformed symmetry [36] on the S-matrix (6.14). The symmetry algebra is the quantum deformation of the universal enveloping algebra, $U_q(\mathfrak{psu}(2|2)\ltimes\mathbb{R}^3)$. The generators of $\mathfrak{psu}(2|2)\ltimes\mathbb{R}^3$ are:[41] $\mathfrak{R}_a^{\ b}$ – generators of one bosonic $\mathfrak{su}(2)$; $\mathfrak{L}_\alpha^{\ \beta}$ – generators of the second bosonic $\mathfrak{su}(2)$; $\mathfrak{Q}_a^{\ \beta}$ – one set of four fermionic generators mixing the two $\mathfrak{su}(2)$s; $\mathfrak{S}_\alpha^{\ b}$ – the second set of four fermionic generators; $\mathfrak{C}$, $\bar{\mathfrak{P}}_\pm$ – the three central charges. The non-trivial (anti-)commutation relations of these generators are

$$\begin{gathered}
[\mathfrak{R}_a^{\ b},\,\mathfrak{R}_c^{\ d}]=\delta_c^b\mathfrak{R}_a^{\ d}-\delta_a^d\mathfrak{R}_c^{\ d}\,,\qquad [\mathfrak{L}_\alpha^{\ \beta},\,\mathfrak{L}_\gamma^{\ \delta}]=\delta_\gamma^\beta\mathfrak{L}_\alpha^{\ \delta}-\delta_\alpha^\delta\mathfrak{L}_\gamma^{\ \delta}\,,\\
[\mathfrak{R}_a^{\ b},\,\mathfrak{Q}_c^{\ \delta}]=\delta_c^b\mathfrak{Q}_a^{\ \delta}-\frac{1}{2}\delta_a^b\mathfrak{Q}_c^{\ \delta}\,,\qquad [\mathfrak{L}_\alpha^{\ \beta},\,\mathfrak{Q}_c^{\ \delta}]=-\delta_\alpha^\delta\mathfrak{Q}_c^{\ \beta}+\frac{1}{2}\delta_\alpha^\beta\mathfrak{Q}_c^{\ \delta}\,,\\
[\mathfrak{R}_a^{\ b},\,\mathfrak{S}_\gamma^{\ d}]=-\delta_a^d\mathfrak{S}_\gamma^{\ b}+\frac{1}{2}\delta_a^b\mathfrak{S}_\gamma^{\ d}\,,\qquad [\mathfrak{L}_\alpha^{\ \beta},\,\mathfrak{S}_\gamma^{\ d}]=\delta_\gamma^\beta\mathfrak{S}_\alpha^{\ d}-\frac{1}{2}\delta_\alpha^\beta\mathfrak{S}_\gamma^{\ d}\\
\{\mathfrak{S}_\alpha^{\ b},\,\mathfrak{Q}_c^{\ \delta}\}=\delta_c^b\mathfrak{L}_\alpha^{\ \delta}+\delta_\alpha^\delta\mathfrak{R}_c^{\ b}+\delta_c^b\delta_\alpha^\delta\mathfrak{C}\,,\\
\{\mathfrak{Q}_a^{\ \beta},\,\mathfrak{Q}_c^{\ \delta}\}=-\epsilon_{ac}\epsilon^{\beta\delta}\bar{\mathfrak{P}}_+\,,\qquad \{\mathfrak{S}_\alpha^{\ b},\,\mathfrak{S}_\gamma^{\ d}\}=\epsilon_{\alpha\gamma}\epsilon^{bd}\bar{\mathfrak{P}}_-\,.
\end{gathered} \tag{6.32}$$

The universal enveloping algebra, $U(\mathfrak{psu}(2|2)\ltimes\mathbb{R}^3)$ is generated by polynomials of the Lie algebra generators. A minimal set of its generators can be taken as follows[42]

$$\begin{array}{lll}
\mathfrak{H}_1=\mathfrak{R}_2^{\ 2}-\mathfrak{R}_1^{\ 1}=2\mathfrak{R}_2^{\ 2}\,, & \mathfrak{E}_1=\mathfrak{R}_2^{\ 1}\,, & \mathfrak{F}_1=\mathfrak{R}_1^{\ 2}\,,\\
\mathfrak{H}_2=-\mathfrak{C}-\frac{1}{2}\mathfrak{H}_1-\frac{1}{2}\mathfrak{H}_3\,, & \mathfrak{E}_2=\mathfrak{S}_4^{\ 2}\,, & \mathfrak{F}_2=\mathfrak{Q}_2^{\ 4}\,,\\
\mathfrak{H}_3=\mathfrak{L}_4^{\ 4}-\mathfrak{L}_3^{\ 3}=2\mathfrak{L}_4^{\ 4}\,, & \mathfrak{E}_3=\mathfrak{L}_4^{\ 3}\,, & \mathfrak{F}_3=\mathfrak{L}_3^{\ 4}\,.
\end{array} \tag{6.33}$$

In this basis the symmetric Cartan matrix is

$$A_{jk}=\begin{pmatrix}2&-1&0\\-1&0&1\\0&1&-2\end{pmatrix} \tag{6.34}$$

[41]Compared to the notation of [36] we have renamed $\mathfrak{S}\leftrightarrow\mathfrak{Q}$ and $\mathfrak{K},\mathfrak{P}\to-\bar{\mathfrak{P}}_+,\bar{\mathfrak{P}}_-$.

[42]Not all the generators of the Lie algebra need to be kept explicitly as certain generators can be rewritten as polynomials of other generators.

and the commutation relations with $\mathfrak{H}_j$ are

$$[\mathfrak{H}_j,\, \mathfrak{H}_k] = 0\,, \qquad [\mathfrak{H}_j,\, \mathfrak{E}_k] = A_{jk}\mathfrak{E}_k\,, \qquad [\mathfrak{H}_j,\, \mathfrak{F}_k] = -A_{jk}\mathfrak{F}_k\,. \tag{6.35}$$

The non-vanishing (anti-)commutators between $\mathfrak{E}_j$ and $\mathfrak{F}_k$ are

$$[\mathfrak{E}_1,\, \mathfrak{F}_1] = \mathfrak{H}_1\,, \qquad \{\mathfrak{E}_2,\, \mathfrak{F}_2\} = -\mathfrak{H}_2\,, \qquad [\mathfrak{E}_3,\, \mathfrak{F}_3] = -\mathfrak{H}_3\,. \tag{6.36}$$

The Serre relations, which we will not give here, are discussed in [36].

In the quantum deformation, the commutation relations with $\mathfrak{H}_j$ stay as they are. The three non-vanishing (anti)-commutators between $\mathfrak{E}_j$ and $\mathfrak{F}_k$ are deformed to

$$[\mathfrak{E}_1,\, \mathfrak{F}_1] = \big[\mathfrak{H}_1\big]_q\,, \qquad \{\mathfrak{E}_2,\, \mathfrak{F}_2\} = -\big[\mathfrak{H}_2\big]_q\,, \qquad [\mathfrak{E}_3,\, \mathfrak{F}_3] = -\big[\mathfrak{H}_3\big]_q\,. \tag{6.37}$$

Here again $\big[x\big]_q \equiv \frac{q^x-q^{-x}}{q-q^{-1}}$ and the quantum deformation parameter q is related to k by (6.13). The Serre relations similarly become quantum-deformed, see [36].

The quantum-deformed generators have the following action on the one-particle states $|\phi_a\rangle$, $|\psi_\alpha\rangle$ [43]

$$\begin{array}{llll}
\mathfrak{H}_1\,|\phi_1\rangle = -\,|\phi_1\rangle\,, & \mathfrak{H}_1\,|\phi_2\rangle = |\phi_2\rangle\,, & \mathfrak{H}_1\,|\psi_3\rangle = 0\,, & \mathfrak{H}_1\,|\psi_4\rangle = 0\,, \\
\mathfrak{H}_2\,|\phi_1\rangle = -(C-\frac{1}{2})\,|\phi_1\rangle\,, & \mathfrak{H}_2\,|\phi_2\rangle = -(C+\frac{1}{2})\,|\phi_2\rangle\,, & \mathfrak{H}_2\,|\psi_3\rangle = -(C-\frac{1}{2})\,|\psi_3\rangle\,, & \mathfrak{H}_2\,|\psi_4\rangle = -(C+\frac{1}{2})\,|\psi_4\rangle\,, \\
\mathfrak{H}_3\,|\phi_1\rangle = 0\,, & \mathfrak{H}_3\,|\phi_2\rangle = 0\,, & \mathfrak{H}_3\,|\psi_3\rangle = -\,|\psi_3\rangle\,, & \mathfrak{H}_3\,|\psi_4\rangle = |\psi_4\rangle\,, \\
\mathfrak{E}_1\,|\phi_1\rangle = q^{\frac{1}{2}}\,|\phi_2\rangle\,, & \mathfrak{E}_1\,|\phi_2\rangle = 0\,, & \mathfrak{E}_1\,|\psi_3\rangle = 0\,, & \mathfrak{E}_1\,|\psi_4\rangle = 0\,, \\
\mathfrak{E}_2\,|\phi_1\rangle = 0\,, & \mathfrak{E}_2\,|\phi_2\rangle = a(\vartheta,k)\,|\psi_4\rangle\,, & \mathfrak{E}_2\,|\psi_3\rangle = b(\vartheta,k)\,|\phi_1\rangle\,, & \mathfrak{E}_2\,|\psi_4\rangle = 0\,, \\
\mathfrak{E}_3\,|\phi_1\rangle = 0\,, & \mathfrak{E}_3\,|\phi_2\rangle = 0\,, & \mathfrak{E}_3\,|\psi_3\rangle = 0\,, & \mathfrak{E}_3\,|\psi_4\rangle = q^{-\frac{1}{2}}\,|\psi_3\rangle\,, \\
\mathfrak{F}_1\,|\phi_1\rangle = 0\,, & \mathfrak{F}_1\,|\phi_2\rangle = q^{-\frac{1}{2}}\,|\phi_1\rangle\,, & \mathfrak{F}_1\,|\psi_3\rangle = 0\,, & \mathfrak{F}_1\,|\psi_4\rangle = 0\,, \\
\mathfrak{F}_2\,|\phi_1\rangle = c(\vartheta,k)\,|\psi_3\rangle\,, & \mathfrak{F}_2\,|\phi_2\rangle = 0\,, & \mathfrak{F}_2\,|\psi_3\rangle = 0\,, & \mathfrak{F}_2\,|\psi_4\rangle = d(\vartheta,k)\,|\phi_1\rangle\,, \\
\mathfrak{F}_3\,|\phi_1\rangle = 0\,, & \mathfrak{F}_3\,|\phi_2\rangle = 0\,, & \mathfrak{F}_3\,|\psi_3\rangle = q^{\frac{1}{2}}\,|\psi_4\rangle\,, & \mathfrak{F}_3\,|\psi_4\rangle = 0\,,
\end{array}$$

$$\mathfrak{C}\,|\Phi_A\rangle = C(\vartheta,k)\,|\Phi_A\rangle\,, \qquad \bar{\mathfrak{P}}_\pm\,|\Phi_A\rangle = \mathcal{P}_\pm(\vartheta,k)\,|\Phi_A\rangle\,. \tag{6.38}$$

The trigonometric relativistic limit of the R-matrix in [36] is found by taking $g\to\infty$. Taking this limit in the functions a, b, c, d, $\mathcal{P}_\pm$ and C given in [36] leads to similar relations as in[44] (4.48),(4.49)

$$a(\vartheta,k) = \sqrt{\frac{1}{2}\sec\frac{\pi}{2k}}\; e^{-\frac{\vartheta}{2}-\frac{i\pi}{4k}}\,, \qquad b(\vartheta,k) = a^*(\vartheta,k)\,, \qquad c(\vartheta,k) = -e^{\vartheta}\;a(\vartheta,k)\,, \qquad d(\vartheta,k) = e^{\vartheta}\;a^*(\vartheta,k)\,, \tag{6.39}$$

$$\mathcal{P}_\pm(\vartheta,k) = \frac{1}{2}\sec\frac{\pi}{2k}\;e^{\pm\vartheta}\,, \qquad C(\vartheta,k) = 0\,. \tag{6.40}$$

The vanishing of C is consistent with (F.8) and confirms the claim that the third central extension vanishes and (5.22) is left as the symmetry algebra. The functions a, b, c, d satisfy the four relations required for the closure of the supersymmetry algebra,

$$ad = \big[C+\frac{1}{2}\big]_q\,, \qquad bc = \big[C-\frac{1}{2}\big]_q\,, \qquad ab = \mathcal{P}_-\,, \qquad cd = -\mathcal{P}_+\,. \tag{6.41}$$

Since $\mathcal{P}_\pm \sim e^{\pm\vartheta}$ we may again interpret $\bar{\mathfrak{P}}_\pm$ as the lightcone momentum symmetry generators up to a normalisation. Once again, the resulting symmetry algebra strongly resembles a 2-d supersymmetry algebra with a global bosonic R-symmetry. The supersymmetry generators are charged under both the Lorentz group (they anticommute to 2-d momentum) and the global bosonic symmetry group. The existence of the global bosonic R-symmetry appears to quantum-deform the supersymmetry. Unlike the reduced $AdS_3\times S^3$ theory here the bosonic symmetry is non-abelian and therefore is non-trivially altered by the quantum deformation.

To define the action of these symmetries on the two particle states, the coproduct is required. In [36] an additional braiding factor, $\mathfrak{U}$, was introduced in the coproduct. In the $g\to\infty$ limit the braiding factor

[43] In appendix F the notation C_i is used in considering two particle states. It stands for the function $C(\vartheta,k)$ evaluated with the rapidity ϑ_i of the first or the second particle.

[44] The $g\to\infty$ limit of b, d, $\mathcal{P}_\pm$ in [36] is slightly technical. The simplest way to compute them is to use a and c in the limit $g\to\infty$ and the relations coming from the closure of the supersymmetry algebra on one-particle states. Alternatively, one can expand out the expressions for b, d, $\mathcal{P}_\pm$ as a power series in g^{-1}. For this one needs the next-to-leading order corrections in g^{-1} for q^C, x^- and x^+.

becomes trivial, see (F.8). The usual quantum-deformed coproduct is

$$\begin{aligned}
&\Delta(\mathfrak{H}_i) = \mathfrak{H}_i \otimes \mathbb{I} + \mathbb{I} \otimes \mathfrak{H}_i\,, && \Delta(\mathfrak{C}) = \mathfrak{C} \otimes \mathbb{I} + \mathbb{I} \otimes \mathfrak{C}\,, \\
&\Delta(\mathfrak{E}_i) = \mathfrak{E}_i \otimes \mathbb{I} + q^{-\mathfrak{H}_i} \otimes \mathfrak{E}_i\,, && \Delta(\bar{\mathfrak{P}}_-) = \bar{\mathfrak{P}}_- \otimes \mathbb{I} + q^{2\mathfrak{C}} \otimes \bar{\mathfrak{P}}_-\,, \\
&\Delta(\mathfrak{F}_i) = \mathfrak{F}_i \otimes q^{\mathfrak{H}_i} + \mathbb{I} \otimes \mathfrak{F}_i\,, && \Delta(\bar{\mathfrak{P}}_+) = \bar{\mathfrak{P}}_+ \otimes q^{-2\mathfrak{C}} + \mathbb{I} \otimes \bar{\mathfrak{P}}_+\,.
\end{aligned} \tag{6.42}$$

The quantum S-matrix (6.14) satisfies the co-commutativity relation with the symmetry generators (which has same form as in (4.46))

$$\Delta_{\rm op}(\mathfrak{J})\ \mathcal{S} = \mathcal{S}\ \Delta(\mathfrak{J})\,. \tag{6.43}$$

6.3 Relating the perturbative S-matrix and the quantum-deformed S-matrix

Let us now discuss a possible relation between the one-loop perturbative S-matrix for the reduced $AdS_5 \times S^5$ theory in its factorised form given in section 6.1.1, and the quantum-deformed S-matrix of [36] in the relativistic trigonometric limit (6.14), (6.15).

Let us look at a particular sector of the quantum-deformed S-matrix, (6.14), (6.17) at leading (tree-level) order in the $1/k$ expansion [45]

$$\begin{array}{cc}
 & \begin{array}{cccc} |\phi_1\phi_1\rangle & |\phi_1\phi_2\rangle & |\phi_2\phi_1\rangle & |\phi_2\phi_2\rangle \end{array} \\
S_q^{(\rm tree)} = & \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 + \frac{i\pi}{2k}\tanh\frac{\theta}{2} & -\frac{i\pi}{k}\coth\theta + \frac{i\pi}{k} & 0 \\
0 & -\frac{i\pi}{k}\coth\theta - \frac{i\pi}{k} & 1 + \frac{i\pi}{2k}\tanh\frac{\theta}{2} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{array}{c} |\phi_1\phi_1\rangle \\ |\phi_1\phi_2\rangle \\ |\phi_2\phi_1\rangle \\ |\phi_2\phi_2\rangle \end{array}
\end{array} \tag{6.44}$$

Compared to the tree-level terms in the perturbative S-matrix (6.11) here the off-diagonal entries contain extra $\pm\frac{i\pi}{k}$ terms. These corrections obey the conjugation relations (6.8) and thus it appears that one does not need to alter the standard commutation relations of creation operators. It is unlikely that such a correction may come from a standard local quartic interaction term (of the type that we considered in [2] and section 4.2).

Adding local counterterms and quantum deforming a symmetry appear to be two disconnected possibilities. Counterterms may restore some properties of integrable theories such as group factorisation. The Yang-Baxter equation, however, has a tree-level anomaly if the global symmetry H is non-abelian and that anomaly cannot be removed by adding a local counterterm. This anomaly can be avoided via a quantum deformation of the symmetry algebra, but how this could happen in a Lagrangian formulation is not clear at the moment.

To relate the two S-matrices we may try an alternative approach – to find a non-unitary rotation of the deformed S-matrix that restores the classical $[SU(2)]^2$ symmetry and maps it into the perturbative S-matrix. Below we find a rotation which does this at tree-level. The rotated S-matrix does not satisfy the unitarity relation, (4.22) or crossing symmetry (4.23) (the crossed S-matrix for the reduced $AdS_5 \times S^5$ theory is given above (6.7)) implying this agreement does not extend beyond the tree level. It is unclear if it is possible to generalise the rotation matrix in such a way that the rotated S-matrix satisfies these physical requirements and agrees with one-loop factorised S-matrix of section 6.1.1.

Whether the quantum-deformed S-matrix or its rotation is actually the physical S-matrix of the perturbative excitations of this reduced theory is an open question.

[45] Note that the phase factor $P_0 = 1 + \mathcal{O}(k^{-2})$ and thus it plays no rôle at the tree level.

6.3.1 Tree-level transformation

To identify a rotation matrix that restores the classical $[SU(2)]^2$ symmetry when applied to the quantum-deformed S-matrix we consider the latter as a matrix acting on the space of two-particle states.

$$S_{q1} = \begin{array}{cc} \begin{matrix} |\phi_1\phi_1\rangle & |\phi_2\phi_2\rangle & |\psi_3\psi_3\rangle & |\psi_4\psi_4\rangle \end{matrix} & \\ \begin{pmatrix} J_1+J_2 & 0 & 0 & 0 \\ 0 & J_1+J_2 & 0 & 0 \\ 0 & 0 & J_3+J_4 & 0 \\ 0 & 0 & 0 & J_3+J_4 \end{pmatrix} & \begin{matrix} |\phi_1\phi_2\rangle \\ |\phi_2\phi_1\rangle \\ |\psi_3\psi_4\rangle \\ |\psi_4\psi_3\rangle \end{matrix} \end{array} \qquad S_{q3} = \begin{array}{cc} \begin{matrix} |\phi_a\psi_\alpha\rangle & |\psi_\alpha\phi_a\rangle \end{matrix} & \\ \begin{pmatrix} J_9 & J_8 \\ J_7 & J_{10} \end{pmatrix} & \begin{matrix} |\phi_a\psi_\alpha\rangle \\ |\psi_\alpha\phi_a\rangle \end{matrix} \end{array}$$

$$S_{q2} = \sec\tfrac{\pi}{k} \begin{array}{cc} \begin{matrix} |\phi_1\phi_2\rangle & |\phi_2\phi_1\rangle & |\psi_3\psi_4\rangle & |\psi_4\psi_3\rangle \end{matrix} & \\ \begin{pmatrix} J_1 & (J_2+J_1)\cos\frac{\pi}{k} - e^{-\frac{i\pi}{k}}J_1 & -J_6 & e^{-\frac{i\pi}{k}}J_6 \\ (J_2+J_1)\cos\frac{\pi}{k} - e^{\frac{i\pi}{k}}J_1 & J_1 & e^{\frac{i\pi}{k}}J_6 & -J_6 \\ -J_5 & e^{-\frac{i\pi}{k}}J_5 & J_3 & (J_4+J_3)\cos\frac{\pi}{k} - e^{-\frac{i\pi}{k}}J_3 \\ e^{\frac{i\pi}{k}}J_5 & -J_5 & (J_4+J_3)\cos\frac{\pi}{k} - e^{\frac{i\pi}{k}}J_3 & J_3 \end{pmatrix} & \begin{matrix} |\phi_1\phi_2\rangle \\ |\phi_2\phi_1\rangle \\ |\psi_3\psi_4\rangle \\ |\psi_4\psi_3\rangle \end{matrix} \end{array} \tag{6.45}$$

Rotating the quantum-deformed S-matrix with the following transformation matrices

$$U_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad U_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$U_2 = \sqrt{\sec\tfrac{\pi}{k}} \begin{pmatrix} \cos\frac{\pi}{2k} & i\sin\frac{\pi}{2k} & 0 & 0 \\ -i\sin\frac{\pi}{2k} & \cos\frac{\pi}{2k} & 0 & 0 \\ 0 & 0 & \cos\frac{\pi}{2k} & i\sin\frac{\pi}{2k} \\ 0 & 0 & -i\sin\frac{\pi}{2k} & \cos\frac{\pi}{2k} \end{pmatrix} \tag{6.46}$$

gives

$$U_1^\dagger S_{q1} U_1 = \begin{array}{cc} \begin{matrix} |\phi_1\phi_1\rangle & |\phi_2\phi_2\rangle & |\psi_3\psi_3\rangle & |\psi_4\psi_4\rangle \end{matrix} & \\ \begin{pmatrix} J_1+J_2 & 0 & 0 & 0 \\ 0 & J_1+J_2 & 0 & 0 \\ 0 & 0 & J_3+J_4 & 0 \\ 0 & 0 & 0 & J_3+J_4 \end{pmatrix} & \begin{matrix} |\phi_1\phi_2\rangle \\ |\phi_2\phi_1\rangle \\ |\psi_3\psi_4\rangle \\ |\psi_4\psi_3\rangle \end{matrix} \end{array} \qquad U_3^\dagger S_{q3} U_3 = \begin{array}{cc} \begin{matrix} |\phi_a\psi_\alpha\rangle & |\psi_\alpha\phi_a\rangle \end{matrix} & \\ \begin{pmatrix} J_9 & J_8 \\ J_7 & J_{10} \end{pmatrix} & \begin{matrix} |\phi_a\psi_\alpha\rangle \\ |\psi_\alpha\phi_a\rangle \end{matrix} \end{array}$$

$$U_2^\dagger S_{q2} U_2 = \begin{array}{cc} \begin{matrix} |\phi_1\phi_2\rangle & |\phi_2\phi_1\rangle & |\psi_3\psi_4\rangle & |\psi_4\psi_3\rangle \end{matrix} & \\ \begin{pmatrix} J_1 & J_2 & -J_6 & J_6 \\ J_2 & J_1 & J_6 & -J_6 \\ -J_5 & J_5 & J_3 & J_4 \\ J_5 & -J_5 & J_4 & J_3 \end{pmatrix} & \begin{matrix} |\phi_1\phi_2\rangle \\ |\phi_2\phi_1\rangle \\ |\psi_3\psi_4\rangle \\ |\psi_4\psi_3\rangle \end{matrix} \end{array} \tag{6.47}$$

The rotation clearly transforms the quantum-deformed S-matrix such that it becomes invariant under the classical symmetry group $[SU(2)]^2$.

The functions J_i parametrise the rotated S-matrix (6.47) in the same way the functions K_i parametrised the one-loop perturbative S-matrix of section 6.1.1, (6.6), (6.4). Furthermore, the leading terms in the expansion of the functions J_i and K_i match (see (6.11) and (6.17)), i.e. the rotated S-matrix (6.47) matches the perturbative S-matrix at the tree-level.

Note that because U_2 is not a unitary matrix there is no contradiction with the fact that while the perturbative S-matrix does not satisfy the YBE at the tree-level, the quantum-deformed S-matrix does by construction.

6.3.2 Beyond tree-level?

Beyond tree-level the functions J_i and K_i disagree. This is expected as the rotated S-matrix (6.47) does not satisfy the unitarity and crossing relations (4.22) and (4.23). In particular, the functions J_i satisfy the crossing relations (6.25), whereas the crossing relations for K_i are given by (6.7).

It is natural to try to generalise the rotation matrices U_k so that the resulting rotated S-matrix still respects the classical group symmetry, $[SU(2)]^2$, but also satisfies unitarity and crossing. Note that this will introduce a rapidity dependence into the rotation matrices U_k.

To get clues about how to do this one may try to find perturbative corrections to the functions J_i restoring unitarity and crossing symmetry order by order in $\frac{1}{k}$. This procedure does not uniquely fix the corrections. The simplest corrections that restore unitarity and crossing at one-loop are exactly those that give the perturbative functions K_i (6.11). At higher orders J_5, J_6, J_7, J_8, J_9 and J_{10} all require corrections, implying that both U_1 and U_3 will also be non-trivial.

To conclude, which transformation relates the perturbative $[SU(2)]^2$ invariant S-matrix to the quantum deformed S-matrix satisfying YBE and what is the origin of the quantum group symmetry remain central open questions. An on-going investigation of the integrable structure [16, 48] and the solitons in similar theories [14, 15, 17] may provide a deeper insight into these issues.

7 Concluding remarks

In this paper we have studied the S-matrix for the perturbative (Lagrangian-field) excitations in a class of 2-d UV finite massive quantum field theories (1.1) which (at least at the classical level) may be interpreted as Pohlmeyer reductions of the $AdS_n \times S^n$ superstring theories.

We reviewed in detail the $AdS_2 \times S^2$ case where the reduced theory is equivalent to the $\mathcal{N} = 2$ supersymmetric sine-Gordon model and thus the exact S-matrix is known (not only in the perturbative-excitation sector but also in the solitonic sectors). Certain properties of this S-matrix (e.g., the structure of the phase factor) are shared by more complicated cases with $n > 2$ where also some new features appear.

In the $AdS_3 \times S^3$ case (which may be viewed as a fermionic generalization of the sum of the complex sine-Gordon and complex sinh-Gordon models) we found that the perturbative one-loop S-matrix derived directly from the Lagrangian formulation requires a correction coming from a particular local counterterm in order to be consistent with quantum integrability, i.e. to satisfy the Yang-Baxter equation. It remains an interesting open question as to whether there is an alternative formulation to (1.1) of this theory (treating bosons and fermions in a more symmetric way – cf. fermionic gauge fields in supersymmetric gauged WZW theory [54, 55]) that automatically produces the required counterterm (4.9) as happened in the complex sine-Gordon case [2]. Such a formulation may also make more manifest a (non-local) 2-d supersymmetry recently observed at the level of the corresponding classical equations of motion [47, 48] and also at the level of the Lagrangian (1.1) in [48]. In particular, non-localities like $(\partial_+)^{-1}$ may be indicating that some massless fields were solved for.

Indeed, we discovered that at the tree level the perturbative S-matrix is invariant under a $(4, 4)$ 2-d supersymmetry algebra originating from the underlying supergroup structure, with the 2-d momentum operator playing the rôle of a central extension. However, in contrast to the case of the $AdS_2 \times S^2$ theory here this tree-level symmetry still requires a non-standard coproduct. The supersymmetry algebra itself gets quantum-deformed, starting at the one-loop level – the anticommutator of the left and right supercharges is modified by a non-linear term depending on the abelian global symmetry generators. It would be important to understand the origin of this quantum deformation, e.g., if it is somehow related to a non-locality of the classical supersymmetry discussed in [47, 48]. Under the assumption of the quantum-deformed supersymmetry algebra we proposed the exact (all orders in $1/k$) expression for the perturbative S-matrix of the reduced $AdS_3 \times S^3$ theory. It remains to study its pole structure and to reconstruct the corresponding solitonic sectors of the full physical S-matrix of this theory.

Our prime interest – the reduced $AdS_5 \times S^5$ theory – has an additional non-trivial feature: the corresponding gauge group $H = [SU(2)]^4$ is non-abelian and consequently the Lagrangian (1.1) does not have an additional global H symmetry as in the abelian case.

The perturbative S-matrix defined by the gauge-fixed Lagrangian has a manifest $H = [SU(2)]^4$ global symmetry which is the global part of the gauge group. We computed the one-loop contribution and have shown that the group factorisation property of the S-matrix discovered at the tree-level in [1] continues to

one-loop order. However, the Yang-Baxter equation which has a tree-level anomaly [1, 2] continues to be violated.

Motivated by the existence of the quantum-deformed supersymmetry in the $AdS_3 \times S^3$ case we proposed that the factorised S-matrix of the $AdS_5 \times S^5$ theory may also have quantum-deformed symmetry while also satisfying the Yang-Baxter equation. We suggested that the corresponding "quantum-deformed" S-matrix may be identified with a quantum-deformed R-matrix corresponding to the symmetries of the $AdS_5 \times S^5$ case, i.e. with the fundamental R-matrix for the quantum deformation of the universal enveloping algebra of $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$. This R-matrix was constructed in [36], and in [37] a particular limit was identified in which its structure becomes similar to that of the tree-level perturbative S-matrix [1] of the reduced $AdS_5 \times S^5$ theory. Here we extended this trigonometric relativistic limit to all orders and demonstrated that the similarity between the resulting R-matrix and the perturbative S-matrix directly defined by the Lagrangian continues to be present also at the subleading one-loop order.

There are, however, significant differences between the quantum-deformed S-matrix and the perturbative S-matrix. In particular, as the algebra of H is non-abelian, its action on the quantum-deformed S-matrix requires a non-trivial coproduct, i.e. is also quantum-deformed. To try to relate the two S-matrices we constructed a non-unitary rotation that mapped one into the other at the tree level. Whether this rotation can be extended to higher orders in $1/k$ is unclear, but if it is possible it will involve introducing a rapidity dependence into the rotation matrix.

The need for this rotation may be related to some conflict between the gauge choice $A_+ = 0$ and the conservation of hidden integrable charges: it is the presence of the global non-abelian symmetry in the gauge-fixed action that leads to a tree-level anomaly in the YBE. This issue, as well the reason for the quantum deformation of the symmetry in the Lagrangian formulation, remain to be clarified.

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Appendix A Complete S-matrices of the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric sine-Gordon models

In this appendix we review the complete S-matrices of the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric sine-Gordon theories [46, 44].

A.1 $\mathcal{N} = 1$ supersymmetric sine-Gordon

The Lagrangian of $\mathcal{N} = 1$ supersymmetric sine-Gordon model may be written as (cf. (3.1))

$$\mathcal{L}^{(sSG)}_{\mathcal{N}=1} = \frac{k}{4\pi}\Big(\partial_+\varphi\partial_-\varphi + \frac{\mu^2}{2}\cos 2\varphi + i\,\delta\partial_-\delta + i\,\nu\partial_+\nu - 2i\mu\,\nu\delta\,\cos\varphi\Big)\,, \tag{A.1}$$

In section 3.3 the S-matrix for the perturbative excitations was reviewed. The complete S-matrix of this theory is much larger due to the existence of solitons and breathers. Schematically, it takes the following form [46]

$$\begin{aligned} \text{soliton-soliton:} \qquad & S_{SG}(\theta,\Delta)\otimes S^{(2)}_{RSG}(\theta,\Delta)\,, \\ \text{soliton-breather:} \qquad & S^{(n)}_{SG}(\theta,\Delta)\otimes S^{(n)}_{RSG}(\theta,\Delta)\,, \\ \text{breather-breather:} \qquad & S^{(n,m)}_{SG}(\theta,\Delta)\otimes S^{(n,m)}_{RSG}(\theta,\Delta)\,, \end{aligned} \tag{A.2}$$

with [46]

$$\Delta = \frac{\pi}{k - \frac{1}{2}} \,. \tag{A.3}$$

The S-matrix factorises into the bosonic (S_{SG}) and supersymmetric (S_{RSG}) parts. The bosonic factor is always the S-matrix for the corresponding excitations in the sine-Gordon model with the coupling (A.3). The supersymmetric part is discussed in [46]. The S-matrix for perturbative excitations we discuss in this paper is a special case of the breather-breather S-matrix with $n = m = 1$, see [46]. The bosonic $n = m = 1$ factor is given by

$$S_{SG}^{(1,1)}(\theta, \Delta) = S_{sg}(\theta, \Delta) = \frac{\sinh\theta + i \sin\Delta}{\sinh\theta - i \sin\Delta} \,. \tag{A.4}$$

Similarly, the supersymmetric $n = m = 1$ S-matrix is given by (3.23)

$$S_{RSG}^{(1,1)}(\theta, \Delta) = S_{\mathcal{N}_1}(\theta, \Delta) \,. \tag{A.5}$$

Looking for poles and zeros on the physical strip amounts to investigating the limit when $\theta = i\Delta$. $S_{sg}(\theta, \Delta)$ has a simple pole at $\theta = i\Delta$. The S and U channels (defined in section 3.3) of $S_{\mathcal{N}_1}(\theta, \Delta)$ have a simple zero, while the T and V channels have no pole or zero. Thus the S and U channels of the total perturbative excitation S-matrix have no pole or zero, while the T and V channels have a simple pole at $\theta = i\Delta$ corresponding to the existence of a bound state.

A.2 $\mathcal{N} = 1$ supersymmetric sinh-Gordon

A related theory is $\mathcal{N} = 1$ supersymmetric sinh-Gordon. This theory is formally related to $\mathcal{N} = 1$ supersymmetric sine-Gordon (A.1) by the transformation

$$\varphi \to i\phi \,, \qquad \nu \to i\rho \,, \qquad \delta \to i\alpha \,, \qquad \text{and} \quad k \to -k \,, \tag{A.6}$$

i.e. the $\mathcal{N} = 1$ supersymmetric sinh-Gordon Lagrangian is (cf. (3.1))

$$\mathcal{L}_{\mathcal{N}=1}^{(sShG)} \quad = \quad k\big(\partial_+\phi\partial_-\phi - \frac{\mu^2}{2}\cosh 2\phi + i\,\alpha\partial_-\alpha + i\,\rho\partial_+\rho - 2i\mu\,\rho\alpha\,\cosh\phi\big) \,. \tag{A.7}$$

The exact S-matrix for the perturbative bosonic and fermionic excitations is related to the above one by $k \to -k$ (see also section 3.3)

$$\begin{gathered} S_{sg}(\theta, -\Delta(k)) \otimes S_{\mathcal{N}_1}(\theta, -\Delta(k)) \,, \\ \Delta(k) = \frac{\pi}{k + \frac{1}{2}} \,. \end{gathered} \tag{A.8}$$

Unlike the $\mathcal{N} = 1$ supersymmetric sine-Gordon case this theory does not have a degenerate vacuum and thus has no soliton solutions. The perturbative excitations are the only physical excitations and thus (A.8) is the complete S-matrix for this theory.

The pole structure of the S-matrix is consistent with this. Looking for poles and and zeros at $\theta = i\Delta$, the S and U channels have neither while the T and V channels have a simple zero. The lack of poles implies hat there are no bound states of the perturbative excitations and is an evidence for the absence of any additional sectors. [47]

A.3 $\mathcal{N} = 2$ supersymmetric sine-Gordon

In section 3.4 the S-matrix for the perturbative excitations of $\mathcal{N} = 2$ supersymmetric sine-Gordon was interpreted [44] as the supersymmetrisation of the bosonic sine-Gordon S-matrix (3.26). One can also interpret the same S-matrix as the supersymmetrisation of the bosonic sinh-Gordon S-matrix. Indeed, rather than labelling the states as in (3.4) let us label them as follows

$$\begin{aligned} |Y\rangle &= |\Phi_{00}\rangle \,, \qquad & |\zeta\rangle &= |\Phi_{01}\rangle \,, \\ |Z\rangle &= |\Phi_{11}\rangle \,, \qquad & |\chi\rangle &= |\Phi_{10}\rangle \,, \end{aligned} \tag{A.9}$$

[46] In the bosonic sine-Gordon theory one has $\Delta = \frac{\pi}{k-1}$.

[47] For reference, it is useful to note that $S_{sg}(\theta, -\Delta)$ has a zero at $\theta = i\Delta$, whilst $S_{\mathcal{N}_1}(\theta, -\Delta)$ has a pole in the S and U channels and neither a pole nor a zero in the T and V channels.

with the index 0 being bosonic and 1 fermionic. Instead of factoring out the S-matrix for the perturbative excitation of the sine-Gordon model we may factor out the S-matrix for the perturbative excitation of the sinh-Gordon model. This corresponds to replacing $k \to -k$ and $\Delta \to -\Delta$ in (3.19) and (3.25). The $\mathcal{N}=2$ supersymmetric sine-Gordon S-matrix then factorises as follows

$$\begin{aligned} S_{SG}(\theta,-\Delta) \otimes S_{\mathcal{N}_1}(\theta,-\Delta) \otimes_{G} S_{\mathcal{N}_1}(\theta,-\Delta)\,, \\ \Delta = \frac{\pi}{k}\,. \end{aligned} \tag{A.10}$$

The two ways of writing this S-matrix are consistent as they have the same poles and zeroes. This can be seen using the results quoted in appendices A.1 and A.2.

While it is possible to factorise the S-matrix as in (A.10) this is not the best way for interpreting it. The reason is that $\mathcal{N}=2$ supersymmetric sine-Gordon model has solitonic excitations which, like in bosonic sine-Gordon case, play the rôle of the elementary excitations in this theory [44]. To construct the S-matrix for these excitations we can generalise the discussion in appendix A.1 for sine-Gordon model, but not for the sinh-Gordon as it has no such excitations. The complete S-matrix for $\mathcal{N}=2$ supersymmetric sine-Gordon theory takes the following schematic form ($\Delta = \frac{\pi}{k}$) [44]

$$\begin{aligned} \text{soliton-soliton:} &\qquad S_{SG}(\theta,\Delta) \otimes S^{(2)}_{RSG}(\theta,\Delta) \otimes_{G} S^{(2)}_{RSG}(\theta,\Delta)\,, \\ \text{soliton-breather:} &\qquad S^{(n)}_{SG}(\theta,\Delta) \otimes S^{(n)}_{RSG}(\theta,\Delta) \otimes_{G} S^{(n)}_{RSG}(\theta,\Delta)\,, \\ \text{breather-breather:} &\qquad S^{(n,m)}_{SG}(\theta,\Delta) \otimes S^{(n,m)}_{RSG}(\theta,\Delta) \otimes_{G} S^{(n,m)}_{RSG}(\theta,\Delta)\,. \end{aligned} \tag{A.11}$$

Identifying the lowest mass $n=m=1$ breather with the perturbative excitation and using (A.4), (A.5) one finds agreement with (3.26).

Appendix B Comments on symmetries of $AdS_3 \times S^3$ superstring and reduced theory

In this appendix we discuss symmetries of the $AdS_3 \times S^3$ reduced and superstring theories. The usual way of constructing the GS superstring sigma model is to start with the coset [53, 5]

$$\frac{[PSU(1,1|2)]^2}{SU(1,1)\times SU(2)}\,. \tag{B.1}$$

The numerator group $[PSU(1,1|2)]^2$ has a bosonic subgroup $[SU(1,1)]^2 \times [SU(2)]^2$ that can be extended by four central elements to $[U(1,1)]^2 \times [U(2)]^2$. The coset (B.1) can then be rewritten as

$$\frac{[U(1,1|2)]^2}{U(1,1)\times U(2)\times [U(1)]^2}\,. \tag{B.2}$$

Following [5] we use the following parametrisation for $[U(1,1|2)]^2$

$$\begin{pmatrix} a_1 & \alpha & 0 & 0 \\ \tilde{\alpha} & b_1 & 0 & 0 \\ 0 & 0 & a_2 & \beta \\ 0 & 0 & \tilde{\beta} & b_2 \end{pmatrix}\,. \tag{B.3}$$

where a_1, a_2 are 2×2 $U(1,1)$ matrices and b_1, b_2 are 2×2 $U(2)$ matrices. α and β are 2×2 complex fermionic matrices, as are $\tilde{\alpha}$ and $\tilde{\beta}$, which contain the same degrees of freedom as α and β respectively.

The corresponding superalgebra $\hat{\mathfrak{f}} = [\mathfrak{u}(1,1|2)]^2$ admits a $\mathbb{Z}_4$ decomposition

$$\hat{\mathfrak{f}} = \hat{\mathfrak{f}}_0 + \hat{\mathfrak{f}}_1 + \hat{\mathfrak{f}}_2 + \hat{\mathfrak{f}}_3\,, \qquad [\hat{\mathfrak{f}}_i, \hat{\mathfrak{f}}_j] \subset \hat{\mathfrak{f}}_{i+j \bmod 4}\,, \tag{B.4}$$

where even/odd subspaces are bosonic/fermionic. The $\mathbb{Z}_4$ decomposition relevant for construction of the reduced $AdS_3 \times S^3$ theory is discussed in [5]; its key property is that it mixes the two copies of $[U(1,1|2)]^2$.

Two of the four central elements discussed above live in $\hat{\mathfrak{f}}_0$ and two in the $\hat{\mathfrak{f}}_2$ subspace.[48] In the $AdS_2 \times S^2$ and $AdS_5 \times S^5$ theories all central elements in $\hat{\mathfrak{f}}_2$ are "manually" projected out. We may do the same for the $AdS_3 \times S^3$ case. For the symmetry analysis of the reduced theory we shall not project out the central elements in $\hat{\mathfrak{f}}_0$. Then the corresponding supercoset we shall consider is

$$\frac{PS([U(1,1|2)]^2)}{U(1,1) \times U(2)} . \tag{B.5}$$

Here P and S in the numerator supergroup correspond to the following constraints on the entries of (B.3)

$$\begin{aligned} &\mathrm{Tr}(a_1) + \mathrm{Tr}(b_1) + \mathrm{Tr}(a_2) + \mathrm{Tr}(b_2) = 0 , \\ &\mathrm{Tr}(a_1) - \mathrm{Tr}(b_1) + \mathrm{Tr}(a_2) - \mathrm{Tr}(b_2) = 0 . \end{aligned} \tag{B.6}$$

Choosing the same element $T \in \mathfrak{a}$ ($\mathfrak{a}$ is the maximal abelian subalgebra of $\hat{\mathfrak{f}}_2$) as in [5], we find that the subalgebra $\mathfrak{h} \subset \hat{\mathfrak{f}}_0$ defined by $[\mathfrak{h}, T] = 0$ is $[\mathfrak{u}(1)]^4$.

One of the four $\mathfrak{u}(1)$s takes the following form

$$\begin{pmatrix} \mathbb{I}_4 & 0 \\ 0 & -\mathbb{I}_4 \end{pmatrix} . \tag{B.7}$$

As the structure of the numerator supergroup (B.3) is block diagonal, any "symmetry" arising from this generator will have a trivial action on all the physical fields in both the superstring and reduced theories. In the reduced $AdS_3 \times S^3$ theory the bosonic H symmetry is therefore $[U(1)]^3$ rather than $[U(1)]^2$ that one might predict by considering just the coset (B.1). The bosonic symmetries of the reduced theory thus consist of a global $[U(1)]^3$ and a gauged $[U(1)]^3$.

It is worth noting that the Lagrangian of the worldsheet superstring is not altered by this discussion. All we have done is to include the central elements in the numerator supergroup and simultaneously divide by them in the denominator group. Therefore, the G-gauge symmetry can always be used to remove them giving back the original supercoset (B.1). The addition of these two central elements does not affect the construction of the reduced theory. They both live in the algebra $\mathfrak{h}$, which is gauged in the reduced theory (i.e. the central elements do not alter the degrees of freedom count).

Starting with the coset (B.5) one may carry out the Pohlmeyer reduction procedure by first gauge-fixing these central elements and then proceed as in [5]. Alternatively, if the central elements are included, then there are two extra degrees of freedom in the group field g and $A_\pm$, but also two extra gauge symmetries.[49] Considering the $A_+ = 0$ gauge, integrating out A_- and using the resulting constraint equation to eliminate ξ one finds that the additional central elements have no effect on the resulting Lagrangian (2.18), i.e. we get again (the quartic expansion of) the Lagrangian constructed in [5].

Appendix C Counterterms from functional determinants

In [38] the one-loop S-matrix for the complex sine-Gordon model was computed using a Lagrangian describing a single complex scalar and the Yang-Baxter equation was found to be violated at this order. The authors proposed that preserving the symmetry (integrability) should be the guiding principle for quantization of the theory and found that there exists a local counterterm which restored the validity of YBE at one-loop order.

It was subsequently suggested [28] that understanding the complex sine-Gordon theory as arising from a gauged WZW [10] should explain the origin of this quantum counterterm. This was demonstrated explicitly in [2] where several methods of deriving the one-loop quantum counterterm for the complex sine-Gordon theory as defined by the gauged WZW plus integrable potential action (cf. the bosonic part of (1.1)) in the path integral were presented, each giving precisely the corrections required to restore YBE and match the expansion of the exact S-matrix of [28]. As was shown in [2], the quantum counterterms may be understood as arising from the functional determinant appearing after integrating out the gauge field $A_\pm$.

[48] This is different from the case of the $AdS_2 \times S^2$ and $AdS_5 \times S^5$ theories, where the numerator supergroup for is of the form $PSU(n, n|2n)$, $n = 1, 2$. Compared to $U(n, n|2n)$ the two central elements have been projected out (the supertrace and the trace). These both live in $\hat{\mathfrak{f}}_2$ when considering the $\mathbb{Z}_4$ decomposition of $U(n, n|2n)$.

[49] One of the $U(1)$ gauge symmetries has trivial action on the group field g, containing the physical bosonic excitations, and on the fermions; however it will have a non-trivial action on the gauge field.

One of the methods discussed in [2] was based on fixing the gauge $A_+ = 0$ and integrating over the gauge field A_- to get a delta function constraint $\delta(g^{-1}\partial_+ g|_{\mathfrak{h}})$ in the path integral. This constraint is then used to eliminate the remaining unphysical degree of freedom ξ in (2.1) from the classical action. In addition, this delta function factor leads to a functional determinant of the form [50]

$$(\det \mathcal{O}_+)^{-1} , \qquad \mathcal{O}_+ v \equiv \partial_+ v + \mathcal{B}_+ v \ , \qquad v \in \mathfrak{h} . \tag{C.1}$$

Introducing an orthonormal basis $\{T_i\}$ for $\mathfrak{h}$ and defining

$$\mathcal{B}_+ T_i = B_{+\,ij} T_j \ , \tag{C.2}$$

one can compute the contribution of this determinant (e.g., via standard anomaly argument). This leads to the following one-loop correction to the Lagrangian

$$\Delta L = -\frac{1}{8\pi} B_{+\,ij} \frac{\partial_-}{\partial_+} B_{+\,ji} + \mathcal{O}(B^3) . \tag{C.3}$$

For the purpose of computing the one-loop two-particle S-matrix higher order $\mathcal{O}(B^3)$ corrections here can be ignored as B is quadratic in the physical field X in (2.1).

C.1 Bosonic theories

Let us briefly review the structure of the resulting corrections for the complex sine-Gordon and the $G/H = SO(N+1)/SO(N)$ generalised sine-Gordon theories [2] using the parametrisation of g given in (2.1). For the complex sine-Gordon case ($G/H = SU(2)/U(1)$)

$$\mathcal{B}_+ v = -\frac{1}{4}\Big([X,\, [\partial_+ X,\, v]] + [\partial_+ X,\, [X,\, v]] \Big) . \tag{C.4}$$

Here $H = U(1)$ so the indices i, j take only a single value and the matrix $B_{+\,ij}$ (C.2) may be denoted simply B_+. Also, B_+ is a total derivative ($B_+ = \partial_+ b$) of a local function of X and thus the correction to the Lagrangian can be written as (we set $B_- = \partial_- b = \frac{\partial_-}{\partial_+} B_+$) [51]

$$\Delta L = -\frac{1}{8\pi} B_+ B_- \ . \tag{C.5}$$

Expanding the field $X = X_m T_m$ where $\{T_m\}$ is an orthonormal basis for $\mathfrak{m}$ (the coset part of the algebra of G) and recalling that we should rescale the fields by $\sqrt{\frac{4\pi}{k}}$, (C.5) precisely matches the result for the local counterterm found by the second method in [2].

For the $G/H = SO(N+1)/SO(N)$ theory we may split $\mathcal{B}_+$ into a symmetric and antisymmetric parts [52]

$$\begin{gathered} \mathcal{B}_+ v = \mathcal{B}^s_+ v + \mathcal{B}^a_+ v \, , \\ \mathcal{B}^s_+ v = -\frac{1}{4}([X,\, [\partial_+ X,\, v]] + [\partial_+ X,\, [X,\, v]]) \, , \qquad \mathcal{B}^a_+ v = \frac{1}{4}[[X,\, \partial_+ X],\, v] \, . \end{gathered} \tag{C.6}$$

As for the complex sine-Gordon case the symmetric part $B^s_{+\,ij}$ (C.2) can be written as a total derivative of a local bilinear expression in X. The antisymmetric part satisfies the following identity [53]

$$(\frac{\partial_-}{\partial_+}\mathcal{B}^a_+) v = \frac{\partial_-}{\partial_+}[[X,\, \partial_+ X],\, v] = -[[X,\, \partial_- X],\, v] + \frac{2}{\partial_+}[[X,\, \partial_+\partial_- X],\, v] \tag{C.7}$$

For the purpose of computing the one-loop S-matrix the second term can be ignored as it vanishes on the linearised equation of motion for X

$$\partial_+\partial_- X + \mu^2 X = 0 \, . \tag{C.8}$$

[50] Here $\mathcal{B}_+$ is a field dependent matrix built out of commutators acting on the vector space $\mathfrak{h}$. In particular, we require that $\mathcal{B}_+ v \in \mathfrak{h}$ for $v \in \mathfrak{h}$. At leading order $\mathcal{B}_+$ is quadratic in fields. For further details see [2].

[51] As $H = U(1)$ is abelian there are no $\mathcal{O}(B^3)$ corrections in (C.3) this case.

[52] To compare to [2] note that $B_{+\,ij} = -\frac{1}{2} f_{mpi} f_{npj} X_m \partial_+ X_n \equiv -\frac{1}{2} V_{mnij} X_m \partial_+ X_n$, where f_{mpi} are the structure constants of the algebra $\mathfrak{g}$ decomposed as $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$.

[53] Here the differential operator $\frac{\partial_-}{\partial_+}$ acts of course only on the fields contained in $\mathcal{B}_+$ and not on v.

We can therefore replace

$$(\frac{\partial_-}{\partial_+}\mathcal{B}^a_+)v = \frac{\partial_-}{\partial_+}[[X,\, \partial_+ X],\, v] \approx -[[X,\, \partial_- X],\, v] \equiv -\mathcal{B}^a_- v\,, \tag{C.9}$$

and thus the correction to the Lagrangian can be written

$$\Delta L = -\frac{1}{8\pi}(B^s_{+\,ij}B^s_{-\,ji} - B^a_{+\,ij}B^a_{-\,ji}) + \mathcal{O}(B^3)\,. \tag{C.10}$$

C.2 Reduced $AdS_n \times S^n$ theories

The method of finding counterterms that worked for the complex sine-Gordon model [2], does not give the required counterterm for the reduced $AdS_3 \times S^3$ theory discussed in section 4.2. Also, the counterterm obtained in the same way as above in the case of the reduced $AdS_5 \times S^5$ theory would break the group factorisation property of the one-loop S-matrix in section 6.1.1.

In this subsection we will postulate a single functional determinant based on the group structure and fields of the theory that gives the required result in all 3 cases: non-trivial one-loop correction in the $AdS_3 \times S^3$ case and no corrections in the $AdS_2 \times S^2$ and the $AdS_5 \times S^5$ cases. [54]

Using integrability as our guiding principle the counterterm (4.9) is required in the reduced $AdS_3 \times S^3$ theory. Below we shall show that it may originate from a particular functional determinant similarly to the bosonic case discussed above. The corresponding operator acts on the superalgebra $\hat{\mathfrak{h}}$ (see section 5.1) whose bosonic subalgebra is $\mathfrak{h}$. If this determinant were to arise in a similar fashion to the bosonic case, there should be analogues of the unphysical degrees of freedom ξ taking values in the fermionic subspaces of $\hat{\mathfrak{h}}$. Comparing to the supersymmetric gauged WZW theory written in components or in superfield forms [54, 55] this suggests that there may be an alternative formulation of the action (1.1) that treats fermions and bosons on a more equal footing and thus requires extra fermionic components of the gauge field.

To define the first-order differential operator whose determinant produces the required contribution let us first recall some group theory of the Pohlmeyer reduction of the $AdS_n \times S^n$ (n = 2, 3, 5) superstring theories (see [3] and section 5.1; for explicit bases and parametrisations see also [5, 20, 1, 21]). Here the constant matrix T that defines the potential of the reduced theory is normalised as $T^2 = -\frac{1}{4}I$. We take an orthonormal basis for the bosonic generators of the superalgebra $\hat{\mathfrak{f}}$. Further we take a basis for the fermionic generators of $\hat{\mathfrak{f}}_1^\perp$ and $\hat{\mathfrak{f}}_3^\perp$, i.e. $T_i^{R^\perp}$ and $T_i^{L^\perp}$, respectively, such that

$$\begin{aligned} \mathrm{STr}(T_i^{R^\perp}T_j^{L^\perp}) &= \delta_{ij}\,, \qquad & \mathrm{STr}(T_i^{L^\perp}T_j^{R^\perp}) &= -\delta_{ij}\,, \\ \mathrm{STr}(T_i^{R^\perp}T_j^{R^\perp}) &= 0\,, \qquad & \mathrm{STr}(T_i^{L^\perp}T_i^{L^\perp}) &= 0\,, \end{aligned} \tag{C.11}$$

where the supertrace is defined as in [3]. [55] It will be useful to denote a basis for the superalgebra $\hat{\mathfrak{h}}$ (a fermionic extension of $\mathfrak{h}$ defined in section 5.1) as T_I. This includes an orthonormal basis for $\mathfrak{h}$ along with the bases for $\hat{\mathfrak{f}}_1^\perp$ and $\hat{\mathfrak{f}}_3^\perp$ described above. We also define the metric

$$\hat{\eta}_{IJ} = \mathrm{STr}(T_I T_J)\,, \qquad \hat{\eta}_{IJ}\hat{\eta}^{JK} = \delta_I^J\,, \qquad \hat{\eta}^{IJ}\hat{\eta}_{JK} = \delta_J^I\,, \tag{C.12}$$

so that $T^I = \hat{\eta}^{IJ}T_J$, $T_I = \hat{\eta}_{IJ}T^J$. This metric is not as simple as in bosonic case due to the more involved supertraces over the fermionic generators (C.11). For the parallel fermionic subspaces $\hat{\mathfrak{f}}_1^\parallel$ and $\hat{\mathfrak{f}}_3^\parallel$ we take similar bases as for the perpendicular fermionic subspaces (C.11), [1]. We also have the following useful identities

$$T_m^{L_{1,2}^\parallel} = 2TT_m^{R_{1,2}^\parallel}\,, \qquad \{T,\, T^{L_{1,2}^\parallel}\} = \{T,\, T^{R_{1,2}^\parallel}\} = 0\,. \tag{C.13}$$

The physical excitations (2.2) live in the subspaces, $X \in \mathfrak{f}_0^\parallel$, $\Psi_{_R} \in \mathfrak{f}_1^\parallel$ and $\Psi_{_R} \in \mathfrak{f}_3^\parallel$. In the reduced $AdS_n \times S^n$ theories each of these subspaces can be split into two halves, both transforming in the same representation of a subgroup of H (see [1] for more details). We take the parametrisation of the physical fields

$$X = Y_m T_m^A + Z_m T_m^S\,, \qquad \Psi_{_R} = \sqrt{i}\zeta_{_R m}T_m^{R_1^\parallel} + \sqrt{i}\chi_{_R m}T_m^{R_2^\parallel}\,, \qquad \Psi_{_L} = \sqrt{i}\zeta_{_L m}T_m^{L_1^\parallel} + \sqrt{i}\chi_{_L m}T_m^{L_2^\parallel}\,. \tag{C.14}$$

[54] No counterterm is needed in the reduced $AdS_2 \times S^2$ theory case to match the exact S-matrix of [43, 44, 45, 46].

[55] The same indices i, j are used here for both fermionic subspaces as well as the bosonic algebra $\mathfrak{h} = \hat{\mathfrak{f}}_0^\perp$ of the previous subsection. This is just a notational convenience and is not meant to indicate that there are the same number of generators in each of these spaces.

As is suggested by the notation the fields Y_m parametrise the part of the reduced theory corresponding to the bosonic AdS_n space and similarly the fields Z_m, the bosonic S^n space.

The functional determinant we propose to consider is ($v \in \hat{\mathfrak{h}}$)

$$(\det \hat{\mathcal{O}}_+)^{-1}\,, \qquad \hat{\mathcal{O}}_+ v \equiv \partial_+ v + \hat{\mathcal{B}}_+ v\,, \qquad \hat{\mathcal{B}}_+ v = \big[[X,\,\partial_+ X] - [\Psi_{_R},\,2T\Psi_{_R}],\; v\big]\,. \tag{C.15}$$

Compared to the operator arising in the bosonic case [2] there is not a part symmetric in X and fermions are included ($\hat{\mathcal{O}}_+$ acts on a superalgebra). The action of $\hat{\mathcal{B}}_+$ on the basis T_I for $\hat{\mathfrak{h}}$ is defined as

$$\hat{\mathcal{B}}_+ T_I = \hat{B}_{+\,I}{}^J T_J\,. \tag{C.16}$$

The linearised equations of motion for X and ψ are (see (2.2))

$$\partial_+\partial_- X + \mu^2 X = 0\,, \quad 2T\partial_+\Psi_{_L} + \mu\Psi_{_R} = 0\,, \quad 2T\partial_-\Psi_{_R} + \mu\Psi_{_L} = 0\,. \tag{C.17}$$

As in the bosonic example we can make use of these equations of motion to replace

$$\Big(\frac{\partial_-}{\partial_+}\hat{\mathcal{B}}_+\Big) v \approx -\big[[X,\,\partial_- X] - [2\Psi_{_L} T,\,\Psi_{_L}],\; v\big] \equiv -\hat{\mathcal{B}}_- v\,. \tag{C.18}$$

The one-loop correction to the Lagrangian coming from this determinant can then be written as [56]

$$\Delta L = \frac{1}{8\pi}\sum_{I,J}(-1)^{[I]}\hat{B}_{+\,I}{}^J \hat{B}_{-\,J}{}^I + \mathcal{O}(\hat{B}^3)\,. \tag{C.19}$$

For the reduced $AdS_2 \times S^2$ and $AdS_5 \times S^5$ theories this correction term vanishes as required. In the case of the $AdS_3 \times S^3$ theory it gives, remarkably, the non-trivial counterterm (4.9) we postulated above to satisfy the YBE. [57]

We should emphasize again that the existence of a single universal expression for the one-loop counterterm is rather non-trivial. Its path integral origin remains, however, to be understood. [58]

Appendix D Factorized S-matrix of reduced $AdS_3 \times S^3$ theory

In this appendix we present the explicit form of the factorised S-matrix (4.19), (4.20) and rewrite it in terms of fields transforming as a vector of the same $SO(2)$ group to enable comparison with the S-matrix of section 2.2.2. This single $SO(2)$ is obtained by identifying the two $SO(2)$s with indices a and α and also the two $SO(2)$s with indices $\dot{a}$ and $\dot{\alpha}$. Naively this gives an $[SO(2)]^2$ symmetry but the actions of these $SO(2)$s coincide, leaving the single $SO(2)$. After identifying the pairs of $SO(2)$s the following rules can be used to translate to the single $SO(2)$ notation

$$\begin{aligned} &\mathbb{I}\otimes\mathbb{I} \to \delta_{mp}\delta_{nq}\,, \qquad &&\mathbb{I}\otimes\mathbb{K} \to \epsilon_{mp}\epsilon_{nq}\,,\\ &\mathbb{K}\otimes\mathbb{I} \to \epsilon_{mp}\epsilon_{nq}\,, \qquad &&\mathbb{K}\otimes\mathbb{K} \to \delta_{mp}\delta_{nq}\,,\\ &(\mathbb{I})_{abcd} = \delta_{ac}\delta_{bd}\,, \qquad &&(\mathbb{K})_{abcd} = \epsilon_{ac}\epsilon_{bd}\,, \end{aligned} \tag{D.1}$$

where the first entry in the tensor product corresponds to undotted indices and the second entry to dotted indices.

The S-matrix has the following structure:

Boson-Boson

$$\begin{aligned} \mathbb{S}\,|Y_m(p_1)Y_n(p_2)\rangle =& \big((L_1^2+L_2^2)\delta_{mp}\delta_{nq} + L_1L_2\epsilon_{mp}\epsilon_{nq}\big)\,|Y_p(p_1)Y_q(p_2)\rangle\\ &+ \big(-2L_9^2(\delta_{mn}\delta_{pq}+\epsilon_{mn}\epsilon_{pq})\big)\,|Z_p(p_1)Z_q(p_2)\rangle \end{aligned}$$

[56] Here we used that $\hat{\eta}^{IL}\hat{\eta}_{KL} = (-1)^{[I]}\delta^I_K$: it equals to 1 if $I=K$ is a bosonic index and to -1 if $I=K$ is a fermionic index. $\hat{\eta}$ was defined in (C.12).

[57] Recall that the physical fields should be rescaled by $\sqrt{\frac{4\pi}{k}}$.

[58] One possibility to include fermions $\Psi_{_R}, \Psi_{_L}$ in the determinant is by a rotation of them by the bosonic field g. Similar rotations were used in the construction of the reduced theory [3, 5]. However, just considering such rotations of fermions in the Lagrangian (1.1) will not produce (C.15): it would be necessary in addition to have extra unphysical degrees of freedom living in the fermionic part of $\hat{\mathfrak{h}}$.

$$
\begin{aligned}
&+\big((L_1+L_2)L_9(\delta_{mn}\delta_{pq}+\epsilon_{mn}\epsilon_{pq})\big)\,|\zeta_p(p_1)\zeta_q(p_2)\rangle\\
&+\big((L_1+L_2)L_9(\delta_{mn}\delta_{pq}+\epsilon_{mn}\epsilon_{pq})\big)\,|\chi_p(p_1)\chi_q(p_2)\rangle\\
\mathbb{S}\,|Z_m(p_1)Z_n(p_2)\rangle =&\big((L_3^2+L_4^2)\delta_{mp}\delta_{nq}+L_3L_4\epsilon_{mp}\epsilon_{nq}\big)\,|Z_p(p_1)Z_q(p_2)\rangle\\
&+\big(-(L_3+L_4)L_{10}(\delta_{mn}\delta_{pq}+\epsilon_{mn}\epsilon_{pq})\big)\,|\chi_p(p_1)\chi_q(p_2)\rangle\\
&+\big(-2L_{10}^2(\delta_{mn}\delta_{pq}+\epsilon_{mn}\epsilon_{pq})\big)\,|Y_p(p_1)Y_q(p_2)\rangle\\
&+\big(-(L_3+L_4)L_{10}(\delta_{mn}\delta_{pq}+\epsilon_{mn}\epsilon_{pq})\big)\,|\zeta_p(p_1)\zeta_q(p_2)\rangle\\
\mathbb{S}\,|Y_m(p_1)Z_n(p_2)\rangle =&\big((L_5^2+L_6^2)\delta_{mp}\delta_{nq}+2L_5L_6\epsilon_{mp}\epsilon_{nq}\big)\,|Y_p(p_1)Z_q(p_2)\rangle\\
&+\big(2L_{11}^2(\delta_{mq}\delta_{pn}+\epsilon_{mq}\epsilon_{pn}\big)\,|Z_p(p_1)Y_q(p_2)\rangle\\
&+\big(-(L_5-L_6)L_{11}(-\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}+\delta_{mq}\delta_{np})\big)\,|\zeta_p(p_1)\chi_q(p_2)\rangle\\
&+\big((L_5-L_6)L_{11}(-\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}+\delta_{mq}\delta_{np})\big)\,|\chi_p(p_1)\zeta_q(p_2)\rangle\\
\mathbb{S}\,|Z_m(p_1)Y_n(p_2)\rangle =&\big((L_7^2+L_8^2)\delta_{mp}\delta_{nq}+2L_7L_8\epsilon_{mp}\epsilon_{nq}\big)\,|Z_p(p_1)Y_q(p_2)\rangle\\
&+\big(2L_{12}^2(\delta_{mq}\delta_{pn}+\epsilon_{mq}\epsilon_{pn}\big)\,|Y_p(p_1)Z_q(p_2)\rangle\\
&+\big((L_7-L_8)L_{12}(-\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}+\delta_{mq}\delta_{np})\big)\,|\chi_p(p_1)\zeta_q(p_2)\rangle\\
&+\big(-(L_7-L_8)L_{12}(-\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}+\delta_{mq}\delta_{np})\big)\,|\zeta_p(p_1)\chi_q(p_2)\rangle
\end{aligned}
$$

Boson-Fermion

$$
\begin{aligned}
\mathbb{S}\,|Y_m(p_1)\zeta_n(p_2)\rangle =&\big((L_1L_5+L_2L_6)\delta_{mp}\delta_{nq}+(L_1L_6+L_2L_5)\epsilon_{mp}\epsilon_{nq}\big)\,|Y_p(p_1)\zeta_q(p_2)\rangle\\
&+\big((L_1-L_2)L_{11}(\delta_{mq}\delta_{pn}+\epsilon_{mq}\epsilon_{pn}\big)\,|\zeta_p(p_1)Y_q(p_2)\rangle\\
&+\big((L_5+L_6)L_9(-\delta_{mq}\delta_{np}+\delta_{mp}\delta_{nq}+\delta_{mn}\delta_{pq})\big)\,|\chi_p(p_1)Z_q(p_2)\rangle\\
\mathbb{S}\,|\zeta_m(p_1)Y_n(p_2)\rangle =&\big((L_1L_7+L_2L_8)\delta_{mp}\delta_{nq}+(L_1L_8+L_2L_7)\epsilon_{mp}\epsilon_{nq}\big)\,|\zeta_p(p_1)Y_q(p_2)\rangle\\
&+\big((L_1-L_2)L_{12}(\delta_{mq}\delta_{pn}+\epsilon_{mq}\epsilon_{pn})\big)\,|Y_p(p_1)\zeta_q(p_2)\rangle\\
&+\big(-(L_7+L_8)L_9(-\delta_{mq}\delta_{np}+\delta_{mp}\delta_{nq}+\delta_{mn}\delta_{pq})\big)\,|Z_p(p_1)\chi_q(p_2)\rangle\\
\mathbb{S}\,|Y_m(p_1)\chi_n(p_2)\rangle =&\big((L_1L_5+L_2L_6)\delta_{mp}\delta_{nq}+(L_1L_6+L_2L_5)\epsilon_{mp}\epsilon_{nq}\big)\,|Y_p(p_1)\chi_q(p_2)\rangle\\
&+\big((L_1-L_2)L_{11}(\delta_{mq}\delta_{pn}+\epsilon_{mq}\epsilon_{pn})\big)\,|\chi_p(p_1)Y_q(p_2)\rangle\\
&+\big(-(L_5+L_6)L_9(-\delta_{mq}\delta_{np}+\delta_{mp}\delta_{nq}+\delta_{mn}\delta_{pq})\big)\,|\zeta_p(p_1)Z_q(p_2)\rangle\\
\mathbb{S}\,|\chi_m(p_1)Y_n(p_2)\rangle =&\big((L_1L_7+L_2L_8)\delta_{mp}\delta_{nq}+(L_1L_8+L_2L_7)\epsilon_{mp}\epsilon_{nq}\big)\,|\chi_p(p_1)Y_q(p_2)\rangle\\
&+\big((L_1-L_2)L_{12}(\delta_{mq}\delta_{pn}+\epsilon_{mq}\epsilon_{pn})\big)\,|Y_p(p_1)\chi_q(p_2)\rangle\\
&+\big((L_7+L_8)L_9(-\delta_{mq}\delta_{np}+\delta_{mp}\delta_{nq}+\delta_{mn}\delta_{pq})\big)\,|Z_p(p_1)\zeta_q(p_2)\rangle\\
\mathbb{S}\,|Z_m(p_1)\zeta_n(p_2)\rangle =&\big((L_3L_7+L_4L_8)\delta_{mp}\delta_{nq}+(L_3L_8+L_4L_7)\epsilon_{mp}\epsilon_{nq}\big)\,|Z_p(p_1)\zeta_q(p_2)\rangle\\
&+\big(-(L_3-L_4)L_{12}(\delta_{mq}\delta_{pn}+\epsilon_{mq}\epsilon_{pn})\big)\,|\zeta_p(p_1)Z_q(p_2)\rangle\\
&+\big((L_7+L_8)L_{10}(-\delta_{mq}\delta_{np}+\delta_{mp}\delta_{nq}+\delta_{mn}\delta_{pq})\big)\,|\chi_p(p_1)Y_q(p_2)\rangle\\
\mathbb{S}\,|\zeta_m(p_1)Z_n(p_2)\rangle =&\big((L_3L_5+L_4L_6)\delta_{mp}\delta_{nq}+(L_3L_6+L_4L_5)\epsilon_{mp}\epsilon_{nq}\big)\,|\zeta_p(p_1)Z_q(p_2)\rangle\\
&+\big(-(L_3-L_4)L_{11}(\delta_{mq}\delta_{pn}+\epsilon_{mq}\epsilon_{pn})\big)\,|Z_p(p_1)\zeta_q(p_2)\rangle\\
&+\big(-(L_5+L_6)L_{10}(-\delta_{mq}\delta_{np}+\delta_{mp}\delta_{nq}+\delta_{mn}\delta_{pq})\big)\,|Y_p(p_1)\chi_q(p_2)\rangle\\
\mathbb{S}\,|Z_m(p_1)\chi_n(p_2)\rangle =&\big((L_3L_7+L_4L_8)\delta_{mp}\delta_{nq}+(L_3L_8+L_4L_7)\epsilon_{mp}\epsilon_{nq}\big)\,|Z_p(p_1)\chi_q(p_2)\rangle\\
&+\big(-(L_3-L_4)L_{12}(\delta_{mq}\delta_{pn}+\epsilon_{mq}\epsilon_{pn})\big)\,|\chi_p(p_1)Z_q(p_2)\rangle\\
&+\big(-(L_7+L_8)L_{10}(-\delta_{mq}\delta_{np}+\delta_{mp}\delta_{nq}+\delta_{mn}\delta_{pq})\big)\,|\zeta_p(p_1)Y_q(p_2)\rangle\\
\mathbb{S}\,|\chi_m(p_1)Z_n(p_2)\rangle =&\big((L_3L_5+L_4L_6)\delta_{mp}\delta_{nq}+(L_3L_6+L_4L_5)\epsilon_{mp}\epsilon_{nq}\big)\,|\chi_p(p_1)Z_q(p_2)\rangle\\
&+\big(-(L_3-L_4)L_{11}(\delta_{mq}\delta_{pn}+\epsilon_{mq}\epsilon_{pn})\big)\,|Z_p(p_1)\chi_q(p_2)\rangle\\
&+\big((L_5+L_6)L_{10}(-\delta_{mq}\delta_{np}+\delta_{mp}\delta_{nq}+\delta_{mn}\delta_{pq})\big)\,|Y_p(p_1)\zeta_q(p_2)\rangle
\end{aligned}
$$

Fermion-Fermion

$$
\begin{aligned}
\mathbb{S}\,|\zeta_m(p_1)\zeta_n(p_2)\rangle =&\big((L_1L_3+L_2L_4)\delta_{mp}\delta_{nq}+(L_1L_4+L_2L_3)\epsilon_{mp}\epsilon_{nq}\big)\,|\zeta_p(p_1)\zeta_q(p_2)\rangle\\
&+\big(2L_9L_{10}(\delta_{mn}\delta_{pq}+\epsilon_{mn}\epsilon_{pq})\big)\,|\chi_p(p_1)\chi_q(p_2)\rangle\\
&+\big((L_1+L_2)L_{10}(\delta_{mn}\delta_{pq}+\epsilon_{mn}\epsilon_{pq})\big)\,|Y_p(p_1)Y_q(p_2)\rangle\\
&+\big(-(L_3+L_4)L_9(\delta_{mn}\delta_{pq}+\epsilon_{mn}\epsilon_{pq})\big)\,|Z_p(p_1)Z_q(p_2)\rangle\\
\mathbb{S}\,|\chi_m(p_1)\chi_n(p_2)\rangle =&\big((L_1L_3+L_2L_4)\delta_{mp}\delta_{nq}+(L_1L_4+L_2L_3)\epsilon_{mp}\epsilon_{nq}\big)\,|\chi_p(p_1)\chi_q(p_2)\rangle\\
&+\big(2L_9L_{10}(\delta_{mn}\delta_{pq}+\epsilon_{mn}\epsilon_{pq})\big)\,|\zeta_p(p_1)\zeta_q(p_2)\rangle\\
&+\big(-(L_3+L_4)L_9(\delta_{mn}\delta_{pq}+\epsilon_{mn}\epsilon_{pq})\big)\,|Z_p(p_1)Z_q(p_2)\rangle\\
&+\big((L_1+L_2)L_{10}(\delta_{mn}\delta_{pq}+\epsilon_{mn}\epsilon_{pq})\big)\,|Y_p(p_1)Y_q(p_2)\rangle\\
\mathbb{S}\,|\zeta_m(p_1)\chi_n(p_2)\rangle =&\big((L_5L_7+L_6L_8)\delta_{mp}\delta_{nq}+(L_5L_8+L_6L_7)\epsilon_{mp}\epsilon_{nq}\big)\,|\zeta_p(p_1)\chi_q(p_2)\rangle\\
&+\big(-L_{11}L_{12}(\delta_{mq}\delta_{pn}+\epsilon_{mq}\epsilon_{pn}\big)\,|\chi_p(p_1)\zeta_q(p_2)\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+\big(-(L_5-L_6)L_{12}(-\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}+\delta_{mq}\delta_{np})\big)\,|Y_p(p_1)Z_q(p_2)\rangle\\
&+\big(-(L_7-L_8)L_{11}(-\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}+\delta_{mq}\delta_{np})\big)\,|Z_p(p_1)Y_q(p_2)\rangle\\
\mathbb{S}\,|\chi_m(p_1)\zeta_n(p_2)\rangle =&\big((L_5L_7+L_6L_8)\delta_{mp}\delta_{nq}+(L_5L_8+L_6L_7)\epsilon_{mp}\epsilon_{nq}\big)\,|\chi_p(p_1)\zeta_q(p_2)\rangle\\
&+\big(-L_{11}L_{12}(\delta_{mq}\delta_{pn}+\epsilon_{mq}\epsilon_{pn}\big)\,|\zeta_p(p_1)\chi_q(p_2)\rangle\\
&+\big((L_7-L_8)L_{11}(-\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}+\delta_{mq}\delta_{np})\big)\,|Z_p(p_1)Y_q(p_2)\rangle\\
&+\big((L_5-L_6)L_{12}(-\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}+\delta_{mq}\delta_{np})\big)\,|Y_p(p_1)Z_q(p_2)\rangle
\end{aligned}
$$

Appendix E Factorized S-matrix of reduced $AdS_5\times S^5$ theory

Similarly to the previous appendix, here we present the factorised S-matrix (6.3), (6.4) in terms of states transforming as a vector of the same $SO(4)$ group to enable comparison with the one-loop S-matrix of section 2.2.3.

Following the discussion of symmetries in section 6.1, the $[SU(2)]^4$ is related to the single $SO(4)$ of sections 6 and 2.2.3 by identifying the two $SU(2)$s with indices a and α and the two $SU(2)$s with indices $\dot{a}$ and $\dot{\alpha}$. The resulting $[SU(2)]^2$ is (locally) the same as $SO(4)$.

After identifying the pairs of $SU(2)$s the following rules can be used to translate to the $SO(4)$ notation

$$
\begin{aligned}
\mathbb{I}\otimes\mathbb{I}&\to\delta_{mp}\delta_{nq}\,, &\quad \mathbb{P}\otimes\mathbb{I}+\mathbb{I}\otimes\mathbb{P}&\to\delta_{mp}\delta_{nq}+\delta_{mq}\delta_{np}-\delta_{mn}\delta_{pq}\,,\\
\mathbb{P}\otimes\mathbb{P}&\to\delta_{mq}\delta_{np}\,, & \mathbb{P}\otimes\mathbb{I}-\mathbb{I}\otimes\mathbb{P}&\to\epsilon_{mnpq}\,,\\
(\mathbb{I})_{ab}^{cd}&=\delta_a^c\delta_b^d\,, & (\mathbb{P})_{ab}^{cd}&=\delta_a^d\delta_b^c\,,
\end{aligned}
\tag{E.1}
$$

where the first entry in the tensor product corresponds to undotted indices and the second to dotted ones.

The S-matrix has the following structure:

Boson-Boson

$$
\begin{aligned}
\mathbb{S}\,|Y_m(p_1)Y_n(p_2)\rangle =&\big((K_1(K_1+K_2))\delta_{mp}\delta_{nq}-K_1K_2\delta_{mn}\delta_{pq}+K_2(K_1+K_2)\delta_{mq}\delta_{np}\big)\,|Y_p(p_1)Y_q(p_2)\rangle\\
&+\big(-K_5^2\delta_{mn}\delta_{pq}\big)\,|Z_p(p_1)Z_q(p_2)\rangle\\
&+\big(-\tfrac{1}{2}(K_1+K_2)K_5(\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}-\delta_{mq}\delta_{np}+\epsilon_{mnpq})+K_2K_5\delta_{mn}\delta_{pq}\big)\,|\zeta_p(p_1)\zeta_q(p_2)\rangle\\
&+\big(-\tfrac{1}{2}(K_1+K_2)K_5(\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}-\delta_{mq}\delta_{np}-\epsilon_{mnpq})+K_2K_5\delta_{mn}\delta_{pq}\big)\,|\chi_p(p_1)\chi_q(p_2)\rangle\\
\mathbb{S}\,|Z_m(p_1)Z_n(p_2)\rangle =&\big((K_3(K_3+K_4))\delta_{mp}\delta_{nq}-K_3K_4\delta_{mn}\delta_{pq}+K_4(K_3+K_4)\delta_{mq}\delta_{np}\big)\,|Z_p(p_1)Z_q(p_2)\rangle\\
&+\big(-K_6^2\delta_{mn}\delta_{pq}\big)\,|Y_p(p_1)Y_q(p_2)\rangle\\
&+\big(\tfrac{1}{2}(K_3+K_4)K_6(\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}-\delta_{mq}\delta_{np}+\epsilon_{mnpq})-K_4K_6\delta_{mn}\delta_{pq}\big)\,|\chi_p(p_1)\chi_q(p_2)\rangle\\
&+\big(\tfrac{1}{2}(K_3+K_4)K_6(\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}-\delta_{mq}\delta_{np}-\epsilon_{mnpq})-K_4K_6\delta_{mn}\delta_{pq}\big)\,|\zeta_p(p_1)\zeta_q(p_2)\rangle\\
\mathbb{S}\,|Y_m(p_1)Z_n(p_2)\rangle =&\big(K_9^2\delta_{mp}\delta_{nq}\big)\,|Y_p(p_1)Z_q(p_2)\rangle\\
&+\big(K_7^2\delta_{mq}\delta_{np}\big)\,|Z_p(p_1)Y_q(p_2)\rangle\\
&+\big(-\tfrac{1}{2}K_7K_9(-\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}+\delta_{mq}\delta_{np}-\epsilon_{mnpq})\big)\,|\zeta_p(p_1)\chi_q(p_2)\rangle\\
&+\big(\tfrac{1}{2}K_7K_9(-\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}+\delta_{mq}\delta_{np}+\epsilon_{mnpq})\big)\,|\chi_p(p_1)\zeta_q(p_2)\rangle\\
\mathbb{S}\,|Z_m(p_1)Y_n(p_2)\rangle =&\big(K_{10}^2\delta_{mp}\delta_{nq}\big)\,|Z_p(p_1)Y_q(p_2)\rangle\\
&+\big(K_8^2\delta_{mq}\delta_{np}\big)\,|Y_p(p_1)Z_q(p_2)\rangle\\
&+\big(\tfrac{1}{2}K_8K_{10}(-\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}+\delta_{mq}\delta_{np}-\epsilon_{mnpq})\big)\,|\chi_p(p_1)\zeta_q(p_2)\rangle\\
&+\big(-\tfrac{1}{2}K_8K_{10}(-\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}+\delta_{mq}\delta_{np}+\epsilon_{mnpq})\big)\,|\zeta_p(p_1)\chi_q(p_2)\rangle
\end{aligned}
$$

Boson-Fermion

$$
\begin{aligned}
\mathbb{S}\,|Y_m(p_1)\zeta_n(p_2)\rangle =&\big((K_1+\tfrac{1}{2}K_2)K_9\delta_{mp}\delta_{nq}+\tfrac{1}{2}K_2K_9(-\delta_{mn}\delta_{pq}+\delta_{mq}\delta_{np}+\epsilon_{mnpq})\big)\,|Y_p(p_1)\zeta_q(p_2)\rangle\\
&+\big(\tfrac{1}{2}K_1K_7(\delta_{mq}\delta_{np}-\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}-\epsilon_{mnpq})+K_2K_7\delta_{mq}\delta_{np}\big)\,|\zeta_p(p_1)Y_q(p_2)\rangle\\
&+\big(-\tfrac{1}{2}K_5K_7(-\delta_{mp}\delta_{nq}+\delta_{mn}\delta_{pq}+\delta_{mq}\delta_{np}+\epsilon_{mnpq})\big)\,|Z_p(p_1)\chi_q(p_2)\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+ \big(-\frac{1}{2}K_5K_9(-\delta_{mq}\delta_{np}+\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}-\epsilon_{mnpq})\big)\left|\chi_p(p_1)Z_q(p_2)\right\rangle \\
\mathbb{S}\left|\zeta_m(p_1)Y_n(p_2)\right\rangle =&\big((K_1+\frac{1}{2}K_2)K_{10}\delta_{mp}\delta_{nq}+\frac{1}{2}K_2K_{10}(-\delta_{mn}\delta_{pq}+\delta_{mq}\delta_{np}+\epsilon_{mnpq})\big)\left|\zeta_p(p_1)Y_q(p_2)\right\rangle \\
&+ \big(\frac{1}{2}K_1K_8(\delta_{mq}\delta_{np}-\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}-\epsilon_{mnpq})+K_2K_8\delta_{mq}\delta_{np}\big)\left|Y_p(p_1)\zeta_q(p_2)\right\rangle \\
&+ \big(\frac{1}{2}K_5K_8(-\delta_{mp}\delta_{nq}+\delta_{mn}\delta_{pq}+\delta_{mq}\delta_{np}+\epsilon_{mnpq})\big)\left|\chi_p(p_1)Z_q(p_2)\right\rangle \\
&+ \big(\frac{1}{2}K_5K_{10}(-\delta_{mq}\delta_{np}+\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}-\epsilon_{mnpq})\big)\left|Z_p(p_1)\chi_q(p_2)\right\rangle \\
\mathbb{S}\left|Y_m(p_1)\chi_n(p_2)\right\rangle =&\big((K_1+\frac{1}{2}K_2)K_9\delta_{mp}\delta_{nq}+\frac{1}{2}K_2K_9(-\delta_{mn}\delta_{pq}+\delta_{mq}\delta_{np}-\epsilon_{mnpq})\big)\left|Y_p(p_1)\chi_q(p_2)\right\rangle \\
&+ \big(\frac{1}{2}K_1K_7(\delta_{mq}\delta_{np}-\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}+\epsilon_{mnpq})+K_2K_7\delta_{mq}\delta_{np}\big)\left|\chi_p(p_1)Y_q(p_2)\right\rangle \\
&+ \big(\frac{1}{2}K_5K_7(-\delta_{mp}\delta_{nq}+\delta_{mn}\delta_{pq}+\delta_{mq}\delta_{np}-\epsilon_{mnpq})\big)\left|Z_p(p_1)\zeta_q(p_2)\right\rangle \\
&+ \big(\frac{1}{2}K_5K_9(-\delta_{mq}\delta_{np}+\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}+\epsilon_{mnpq})\big)\left|\zeta_p(p_1)Z_q(p_2)\right\rangle \\
\mathbb{S}\left|\chi_m(p_1)Y_n(p_2)\right\rangle =&\big((K_1+\frac{1}{2}K_2)K_{10}\delta_{mp}\delta_{nq}+\frac{1}{2}K_2K_{10}(-\delta_{mn}\delta_{pq}+\delta_{mq}\delta_{np}-\epsilon_{mnpq})\big)\left|\chi_p(p_1)Y_q(p_2)\right\rangle \\
&+ \big(\frac{1}{2}K_1K_8(\delta_{mq}\delta_{np}-\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}+\epsilon_{mnpq})+K_2K_8\delta_{mq}\delta_{np}\big)\left|Y_p(p_1)\chi_q(p_2)\right\rangle \\
&+ \big(-\frac{1}{2}K_5K_8(-\delta_{mp}\delta_{nq}+\delta_{mn}\delta_{pq}+\delta_{mq}\delta_{np}-\epsilon_{mnpq})\big)\left|\zeta_p(p_1)Z_q(p_2)\right\rangle \\
&+ \big(-\frac{1}{2}K_5K_{10}(-\delta_{mq}\delta_{np}+\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}+\epsilon_{mnpq})\big)\left|Z_p(p_1)\zeta_q(p_2)\right\rangle \\
\mathbb{S}\left|Z_m(p_1)\zeta_n(p_2)\right\rangle =&\big((K_3+\frac{1}{2}K_4)K_{10}\delta_{mp}\delta_{nq}+\frac{1}{2}K_4K_{10}(-\delta_{mn}\delta_{pq}+\delta_{mq}\delta_{np}-\epsilon_{mnpq})\big)\left|Z_p(p_1)\zeta_q(p_2)\right\rangle \\
&+ \big(-\frac{1}{2}K_3K_8(\delta_{mq}\delta_{np}-\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}+\epsilon_{mnpq})-K_4K_8\delta_{mq}\delta_{np}\big)\left|\zeta_p(p_1)Z_q(p_2)\right\rangle \\
&+ \big(\frac{1}{2}K_6K_8(-\delta_{mp}\delta_{nq}+\delta_{mn}\delta_{pq}+\delta_{mq}\delta_{np}-\epsilon_{mnpq}\big)\left|Y_p(p_1)\chi_q(p_2)\right\rangle \\
&+ \big(-\frac{1}{2}K_6K_{10}(-\delta_{mq}\delta_{np}+\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}+\epsilon_{mnpq})\big)\left|\chi_p(p_1)Y_q(p_2)\right\rangle \\
\mathbb{S}\left|\zeta_m(p_1)Z_n(p_2)\right\rangle =&\big((K_3+\frac{1}{2}K_4)K_9\delta_{mp}\delta_{nq}+\frac{1}{2}K_4K_9(-\delta_{mn}\delta_{pq}+\delta_{mq}\delta_{np}-\epsilon_{mnpq})\big)\left|\zeta_p(p_1)Z_q(p_2)\right\rangle \\
&+ \big(-\frac{1}{2}K_3K_7(\delta_{mq}\delta_{np}-\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}+\epsilon_{mnpq})-K_4K_7\delta_{mq}\delta_{np}\big)\left|Z_p(p_1)\zeta_q(p_2)\right\rangle \\
&+ \big(-\frac{1}{2}K_6K_7(-\delta_{mp}\delta_{nq}+\delta_{mn}\delta_{pq}+\delta_{mq}\delta_{np}-\epsilon_{mnpq}\big)\left|\chi_p(p_1)Y_q(p_2)\right\rangle \\
&+ \big(\frac{1}{2}K_6K_9(-\delta_{mq}\delta_{np}+\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}+\epsilon_{mnpq})\big)\left|Y_p(p_1)\chi_q(p_2)\right\rangle \\
\mathbb{S}\left|Z_m(p_1)\chi_n(p_2)\right\rangle =&\big((K_3+\frac{1}{2}K_4)K_{10}\delta_{mp}\delta_{nq}+\frac{1}{2}K_4K_{10}(-\delta_{mn}\delta_{pq}+\delta_{mq}\delta_{np}+\epsilon_{mnpq})\big)\left|Z_p(p_1)\chi_q(p_2)\right\rangle \\
&+ \big(-\frac{1}{2}K_3K_8(\delta_{mq}\delta_{np}-\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}-\epsilon_{mnpq})-K_4K_8\delta_{mq}\delta_{np}\big)\left|\chi_p(p_1)Z_q(p_2)\right\rangle \\
&+ \big(-\frac{1}{2}K_6K_8(-\delta_{mp}\delta_{nq}+\delta_{mn}\delta_{pq}+\delta_{mq}\delta_{pn}+\epsilon_{mnpq}\big)\left|Y_p(p_1)\zeta_q(p_2)\right\rangle \\
&+ \big(\frac{1}{2}K_6K_{10}(-\delta_{mq}\delta_{np}+\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}-\epsilon_{mnpq})\big)\left|\zeta_p(p_1)Y_q(p_2)\right\rangle \\
\mathbb{S}\left|\chi_m(p_1)Z_n(p_2)\right\rangle =&\big((K_3+\frac{1}{2}K_4)K_9\delta_{mp}\delta_{nq}+\frac{1}{2}K_4K_9(-\delta_{mn}\delta_{pq}+\delta_{mq}\delta_{np}+\epsilon_{mnpq})\big)\left|\chi_p(p_1)Z_q(p_2)\right\rangle \\
&+ \big(-\frac{1}{2}K_3K_7(\delta_{mq}\delta_{np}-\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}-\epsilon_{mnpq})-K_4K_7\delta_{mq}\delta_{np}\big)\left|Z_p(p_1)\chi_q(p_2)\right\rangle \\
&+ \big(\frac{1}{2}K_6K_7(-\delta_{mp}\delta_{nq}+\delta_{mn}\delta_{pq}+\delta_{mq}\delta_{np}+\epsilon_{mnpq}\big)\left|\zeta_p(p_1)Y_q(p_2)\right\rangle \\
&+ \big(-\frac{1}{2}K_6K_9(-\delta_{mq}\delta_{np}+\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}-\epsilon_{mnpq})\big)\left|Y_p(p_1)\zeta_q(p_2)\right\rangle
\end{aligned}
$$

Fermion-Fermion

$$
\begin{aligned}
\mathbb{S}\left|\zeta_m(p_1)\zeta_n(p_2)\right\rangle =&\big(-\frac{1}{2}(K_1K_4-K_2K_3)\epsilon_{mnpq}+(K_1K_3+\frac{1}{2}(K_1K_4+K_2K_3))\delta_{mp}\delta_{nq} \\
&\quad -\frac{1}{2}(K_1K_4+K_2K_3)\delta_{mn}\delta_{pq}+(K_2K_4+\frac{1}{2}(K_1K_4+K_2K_3))\delta_{mq}\delta_{np}\big)\left|\zeta_p(p_1)\zeta_q(p_2)\right\rangle \\
&+ \big(K_5K_6\delta_{mn}\delta_{pq}\big)\left|\chi_p(p_1)\chi_q(p_2)\right\rangle \\
&+ \big(-\frac{1}{2}(K_1+K_2)K_6(\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}-\delta_{mq}\delta_{np}+\epsilon_{mnpq})+K_2K_6\delta_{mn}\delta_{pq}\big)\left|Y_p(p_1)Y_q(p_2)\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+ \big(\tfrac{1}{2}(K_3+K_4)K_5(\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}-\delta_{mq}\delta_{np}-\epsilon_{mnpq})-K_4K_5\delta_{mn}\delta_{pq}\big)\,|Z_p(p_1)Z_q(p_2)\rangle \\
\mathbb{S}\,|\chi_m(p_1)\chi_n(p_2)\rangle =& \big(\tfrac{1}{2}(K_1K_4-K_2K_3)\epsilon_{mnpq}+(K_1K_3+\tfrac{1}{2}(K_1K_4+K_2K_3))\delta_{mp}\delta_{nq} \\
&-\tfrac{1}{2}(K_1K_4+K_2K_3)\delta_{mn}\delta_{pq}+(K_2K_4+\tfrac{1}{2}(K_1K_4+K_2K_3))\delta_{mq}\delta_{np}\big)\,|\chi_p(p_1)\chi_q(p_2)\rangle \\
&+ \big(K_5K_6\delta_{mn}\delta_{pq}\big)\,|\zeta_p(p_1)\zeta_q(p_2)\rangle \\
&+ \big(\tfrac{1}{2}(K_3+K_4)K_5(\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}-\delta_{mq}\delta_{np}+\epsilon_{mnpq})-K_4K_5\delta_{mn}\delta_{pq}\big)\,|Z_p(p_1)Z_q(p_2)\rangle \\
&+ \big(-\tfrac{1}{2}(K_1+K_2)K_6(\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}-\delta_{mq}\delta_{np}-\epsilon_{mnpq})+K_2K_6\delta_{mn}\delta_{pq}\big)\,|Y_p(p_1)Y_q(p_2)\rangle \\
\mathbb{S}\,|\zeta_m(p_1)\chi_n(p_2)\rangle =& \big(K_9K_{10}\delta_{mp}\delta_{nq}\big)\,|\zeta_p(p_1)\chi_q(p_2)\rangle \\
&+ \big(-K_7K_8\delta_{mq}\delta_{np}\big)\,|\chi_p(p_1)\zeta_q(p_2)\rangle \\
&+ \big(-\tfrac{1}{2}K_8K_9(-\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}+\delta_{mq}\delta_{np}-\epsilon_{mnpq})\big)\,|Y_p(p_1)Z_q(p_2)\rangle \\
&+ \big(-\tfrac{1}{2}K_7K_{10}(-\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}+\delta_{mq}\delta_{np}+\epsilon_{mnpq})\big)\,|Z_p(p_1)Y_q(p_2)\rangle \\
\mathbb{S}\,|\chi_m(p_1)\zeta_n(p_2)\rangle =& \big(K_9K_{10}\delta_{mp}\delta_{nq}\big)\,|\chi_p(p_1)\zeta_q(p_2)\rangle \\
&+ \big(-K_7K_8\delta_{mq}\delta_{np}\big)\,|\zeta_p(p_1)\chi_q(p_2)\rangle \\
&+ \big(\tfrac{1}{2}K_7K_{10}(-\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}+\delta_{mq}\delta_{np}-\epsilon_{mnpq})\big)\,|Z_p(p_1)Y_q(p_2)\rangle \\
&+ \big(\tfrac{1}{2}K_8K_9(-\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}+\delta_{mq}\delta_{np}+\epsilon_{mnpq})\big)\,|Y_p(p_1)Z_q(p_2)\rangle
\end{aligned}
$$

Appendix F Trigonometric relativistic limit of quantum-deformed $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ R-matrix

In [36] the fundamental R-matrix of the quantum deformation of the centrally extended superalgebra $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ was constructed. This R-matrix depends on various parameters: the global algebra parameters, α, g; the quantum group deformation parameter q; and the spectral parameters x_i^+, x_i^-, γ_i ($i = 1, 2$), where x_i^+ and x_i^- are related by a constraint equation. There is a natural choice for the parameters γ_i (which effectively control the normalisation of the fermions) that is given in [36]. We take γ_i to be given by this choice, rescaled by a factor of $\sqrt[4]{-g^2}$.

In this section we generalise the limit of [37] that leads to a trigonometric, relativistic, q-deformed, classical r-matrix giving the exact trigonometric, relativistic, q-deformed R-matrix. The classical limit investigated in [37] corresponds to expanding the quantum deformation parameter,

$$
q = 1 + \frac{h}{2g} + \mathcal{O}(g^{-2})\,. \tag{F.1}
$$

g^{-1} is playing the rôle of $\hbar$, i.e. it is small but finite. The relativistic trigonometric limit that is relevant for the reduced $AdS_5 \times S^5$ theory is $h \to \infty$ (discussed in section 6.5 of [37]).

To generalise it we set

$$
h \propto \frac{g}{k}\,, \tag{F.2}
$$

and take k^{-1} to be the parameter playing the rôle of $\hbar$. Assuming q (or equivalently k) is finite, the strict $h \to \infty$ limit corresponds to the strict $g \to \infty$ limit in the new variables (g, k).

All dependence on α in the R-matrix then drops out (we just set α equal to one). The spectral parameters x_i^+, x_i^- are reinterpreted in terms of rapidities, and the quantum deformation parameter q, is parametrised in terms of the coupling k as

$$
q = 1 + \sum_{n=1}^{\infty} \frac{a_n}{k^n}\,. \tag{F.3}
$$

This limit is essentially the same as the one described in section 6.5 of [37], rewritten in a way that allows us to consider it to higher orders than just leading (tree-level) one. The fixing of the remaining parameters is done to match the resulting R-matrix with our one-loop perturbation theory result for the perturbative S-matrix as closely as possible.

In the strict $g \to \infty$ limit the constraint equation relating $x_i^\pm$ reduces to

$$(x_i^+)^2 = q^2 (x_i^-)^2 \,. \tag{F.4}$$

To solve it we set

$$x_i^\pm = -iq^{\pm\frac{1}{2}}\, e^{-\vartheta_i} \,. \tag{F.5}$$

As suggested by this ansatz, the variables ϑ_i are identified with the rapidities. The matching to the one-loop S-matrix as closely as possible suggests the following expansion of q to one-loop order

$$q = 1 - \frac{i\pi}{k} - \frac{\pi^2}{2k^2} + \mathcal{O}(k^{-3}) \,. \tag{F.6}$$

This prompts us to make a conjecture that the exact form of q should be

$$q = \exp\Big(- \frac{i\pi}{k} \Big) \,. \tag{F.7}$$

For convenience we give some of the quantities that appear in the R-matrix of [36] in this limit and parametrisation,

$$\gamma_i = \frac{1}{\sqrt{2\cos\frac{\pi}{2k}}}\, e^{-\frac{\vartheta_i}{2} - \frac{i\pi}{4k}} \,, \qquad q^{C_i} = U_i = 1 \,. \tag{F.8}$$

The R-matrix is then parametrised by ten functions J_i,[59]

$$\begin{aligned}
\mathcal{R}\,|\phi_1\phi_1\rangle =& \big(J_1 + J_2\big)\,|\phi_1\phi_1\rangle \\
\mathcal{R}\,|\phi_1\phi_2\rangle =& J_1 \sec\frac{\pi}{k}\,|\phi_1\phi_2\rangle + \big(J_2 - iJ_1\tan\frac{\pi}{k}\big)\,|\phi_2\phi_1\rangle - J_5\sec\frac{\pi}{k}\,|\psi_3\psi_4\rangle + J_5(1 + i\tan\frac{\pi}{k})\,|\psi_4\psi_3\rangle \\
\mathcal{R}\,|\phi_2\phi_1\rangle =& J_1 \sec\frac{\pi}{k}\,|\phi_2\phi_1\rangle + \big(J_2 + iJ_1\tan\frac{\pi}{k}\big)\,|\phi_1\phi_2\rangle - J_5\sec\frac{\pi}{k}\,|\psi_4\psi_3\rangle + J_5(1 - i\tan\frac{\pi}{k})\,|\psi_3\psi_4\rangle \\
\mathcal{R}\,|\phi_2\phi_2\rangle =& \big(J_1 + J_2\big)\,|\phi_2\phi_2\rangle \\
\mathcal{R}\,|\psi_3\psi_3\rangle =& \big(J_3 + J_4\big)\,|\psi_3\psi_3\rangle \\
\mathcal{R}\,|\psi_3\psi_4\rangle =& J_3 \sec\frac{\pi}{k}\,|\psi_3\psi_4\rangle + \big(J_4 - iJ_3\tan\frac{\pi}{k}\big)\,|\psi_4\psi_3\rangle - J_6\sec\frac{\pi}{k}\,|\phi_1\phi_2\rangle + J_6(1 + i\tan\frac{\pi}{k})\,|\phi_2\phi_1\rangle \\
\mathcal{R}\,|\psi_4\psi_3\rangle =& J_3 \sec\frac{\pi}{k}\,|\psi_4\psi_3\rangle + \big(J_4 + iJ_3\tan\frac{\pi}{k}\big)\,|\psi_3\psi_4\rangle - J_6\sec\frac{\pi}{k}\,|\phi_2\phi_1\rangle + J_6(1 - i\tan\frac{\pi}{k})\,|\phi_1\phi_2\rangle \\
\mathcal{R}\,|\psi_4\psi_4\rangle =& \big(J_3 + J_4\big)\,|\psi_4\psi_4\rangle \\
\mathcal{R}\,|\phi_a\psi_\beta\rangle =& J_7\;\delta_a^d\delta_\beta^\gamma\,|\psi_\gamma\phi_d\rangle + J_9\;\delta_a^c\delta_\beta^\delta\,|\phi_c\psi_\delta\rangle \\
\mathcal{R}\,|\psi_\alpha\phi_b\rangle =& J_8\;\delta_\alpha^\delta\delta_b^c\,|\phi_c\psi_\delta\rangle + J_{10}\;\delta_\alpha^\gamma\delta_b^d\,|\psi_\gamma\phi_d\rangle
\end{aligned} \tag{F.9}$$

As in sections 6.1.1 and 6.2 our index notation in as follows

$$a = (1,\,2)\,, \quad \alpha = (3,\,4)\,, \quad A = (a,\,\alpha)\,, \qquad \text{with the fermionic grading} \quad \big[a\big] = 0\,, \quad \big[\alpha\big] = 1\,. \tag{F.10}$$

In the trigonometric relativistic limit the functions J_i are given in (6.15). The phase factor R_0 of [36] is related to P_0 by

$$R_0(\theta,k) = P_0(\theta,k)\,\mathrm{cosech}\,\frac{\theta}{2}\,\sinh\Big(\frac{\theta}{2} + \frac{i\pi}{2k}\Big) \,. \tag{F.11}$$

The requirement of unitarity of the R-matrix is [36]

$$\mathcal{R}_{12}\mathcal{R}_{21} = \mathbb{I}\otimes\mathbb{I} \,. \tag{F.12}$$

In terms of the rapidities the interchange of 1 and 2 sends $\theta \equiv \vartheta_1 - \vartheta_2 \to -\theta$. The R-matrix satisfies (F.12) so long as the phase factor satisfies

$$R_0(\theta,k)R_0(-\theta,k) = 1 \,. \tag{F.13}$$

[59]The ten functions J_I are related to the ten functions $A, B, \ldots, K$ of [36] as follows

$$(J_1, J_2, J_3, J_4, J_5, J_6, J_7, J_8, J_9, J_{10}) = \Big(\frac{A-B}{2}, \frac{A+B}{2}, -\frac{D-E}{2}, -\frac{D+E}{2}, -\frac{C}{2}, \frac{F}{2}, H, K, G, L\Big) \,.$$

We have also renamed $\psi_{1,2} \to \psi_{3,4}$.

This implies the following constraint on the phase P_0 in (F.11)

$$P_0(\theta,k)P_0(-\theta,k)=\frac{\sinh^2\frac{\theta}{2}}{\sinh^2\frac{\theta}{2}+\sin^2\frac{\pi}{2k}}\,. \tag{F.14}$$

The quantum-deformed R-matrix also has a crossing symmetry subject to a constraint on the phase factor. This crossing symmetry is given by [36]

$$(\mathcal{C}^{-1}\otimes\mathbb{I})\mathcal{R}_{\bar{1}2}^{\mathrm{ST}\otimes\mathbb{I}}(\mathbb{I}\otimes\mathcal{C})\mathcal{R}_{12}=\mathbb{I}\otimes\mathbb{I}\,. \tag{F.15}$$

ST is supertransposition and the action of charge conjugation $\mathcal{C}$ on one-particle states is

$$\mathcal{C}\,|\phi_1\rangle=-q^{\frac12}\,|\phi_2\rangle\,,\qquad \mathcal{C}\,|\psi_3\rangle=-q^{\frac12}\,|\psi_4\rangle\,,\qquad \mathcal{C}\,|\phi_2\rangle=q^{-\frac12}\,|\phi_1\rangle\,,\qquad \mathcal{C}\,|\psi_4\rangle=q^{-\frac12}\,|\psi_3\rangle\,. \tag{F.16}$$

In the $g\to\infty$ limit the crossed spectral parameters are given by

$$\bar{x}_i^{\pm}=-x_i^{\pm}\,,\quad \bar{\gamma}_i=-i\gamma_i\,,\qquad q^{\bar{C}_i}=\bar{U}_i=1\,. \tag{F.17}$$

The crossing symmetry relates the R-matrices $\mathcal{R}_{12}$ and $\mathcal{R}_{\bar{1}2}$. Considering the "Lorentz invariant" combinations of the spectral parameters, e.g., x_1/x_2, and comparing to $\bar{x}_1/x_2$, we see that these are related by $\theta\equiv\vartheta_1-\vartheta_2\to\theta+i\pi$. The R-matrix satisfies the crossing relation (F.15) if the phase factor satisfies

$$R_0(\theta,k)\;R_0(\theta+i\pi,k)=\frac{\cosh\left(\frac{\theta}{2}+\frac{i\pi}{2k}\right)}{\sinh\left(\frac{\theta}{2}-\frac{i\pi}{2k}\right)}\tanh\frac{\theta}{2}\;. \tag{F.18}$$

Combining with the unitarity relation, (F.14) then implies the following constraint on the phase P_0 (F.11)

$$P_0(i\pi-\theta,k)=P_0(\theta,k)\,. \tag{F.19}$$

The crossing symmetry also implies the relations (6.25) between the functions J_i.

The conjugation relations (6.26) hold as long as the phase factor satisfies

$$R_0(\theta,k)=R_0^*(-\theta,k)\qquad\Rightarrow\qquad P_0(\theta,k)=P_0^*(-\theta,k)\,. \tag{F.20}$$

These relations are equivalent to the matrix unitarity relations of [36],

$$(\mathcal{R}_{12})^\dagger\mathcal{R}_{12}=\mathbb{I}\otimes\mathbb{I}\,. \tag{F.21}$$

Appendix G One-loop S-matrix of $SO(N+1)/SO(N)$ generalized sine-Gordon models

Here we shall discuss the one-loop S-matrix of the bosonic $G/H=SO(N+1)/SO(N)$ gauged WZW model deformed by an integrable potential as in (1.1). This one-loop S-matrix, including the determinant corrections from integrating out the unphysical fields, was computed in [2]. Here we shall make some additional comments and clarify the structure of the result.

In the $N=4$ case the determinant corrections result in the S-matrix factorising under $\mathfrak{so}(4)\cong\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ at one-loop [2]. Below we will see that, like in the reduced $AdS_5\times S^5$ theory, one can find a quantum-deformed S-matrix similar to the one-loop factorised S-matrix that satisfies the Yang-Baxter equation.

The Lagrangian found by fixing the $A_+=0$ gauge and eliminating A_- and ξ is given by taking (2.3) with $Y=\zeta=\chi=0$ and m,n,p,q now being $SO(N)$ vector indices

$$\begin{aligned}\mathcal{L}_b \;=\;& \frac12\partial_+X_m\partial_-X_m-\frac{\mu^2}{2}X_mX_m\\ &+\frac{\pi}{k}\Big[\frac13X_mX_m\partial_+X_n\partial_-X_n-\frac13X_m\partial_+X_mX_n\partial_-X_n+\frac{\mu^2}{6}X_mX_mX_nX_n\Big]+\mathcal{O}(k^{-2})\,.\end{aligned} \tag{G.1}$$

The one-loop S-matrix for this theory [2] is given by the usual Feynman diagram contribution plus the contribution of the determinant obtained upon integrating out A_- and ξ [60]

$$\mathbb{S}\,|X_m(p_1)X_n(p_2)\rangle = \Big(S_1(\theta,k)\delta_{mn}\delta_{pq} + S_2(\theta,k)\delta_{mp}\delta_{nq} + S_3(\theta,k)\delta_{mq}\delta_{np}\Big)\,|X_p(p_1)X_q(p_2)\rangle \ , \tag{G.2}$$

$$S_i = \bar{S}_i + \Delta S_i \ , \tag{G.3}$$

$$\begin{aligned}\bar{S}_3(\theta,k) =& \bar{S}_1(i\pi-\theta,k) = \frac{i\pi}{k}\coth\theta + \frac{i\pi}{2k^2}(\operatorname{cosech}\theta - \coth\theta) \\ & - \frac{\pi^2}{k^2}\coth\theta\operatorname{cosech}\theta + \frac{i\pi}{2k^2}(N-2)\theta\coth^2\theta + \mathcal{O}(k^{-3})\end{aligned} \tag{G.4}$$

$$\bar{S}_2(\theta,k) = 1 + \frac{i\pi}{k}\operatorname{cosech}\theta - \frac{\pi^2}{k^2}(\frac{1}{2} + \operatorname{cosech}^2\theta) + \frac{i\pi}{2k^2}(N-2)\operatorname{cosech}\theta + \mathcal{O}(k^{-3})$$

$$\begin{aligned}\Delta S_3(\theta,k) = \Delta S_1(i\pi-\theta,k) =& -\frac{i\pi}{2k^2}(\operatorname{cosech}\theta - \coth\theta) - (N-2)\frac{i\pi}{k^2}\coth\theta \\ \Delta S_2(\theta,k) =& -(N-2)\frac{i\pi}{k^2}\operatorname{cosech}\theta\end{aligned} \tag{G.5}$$

The correction ΔS_k coming from the determinant (or the corresponding one-loop counterterm which it produces) splits into two parts. The part not proportional to $(N-2)$ is required to maintain some of the consequences of integrability: in the abelian H case, $N=2$, the corresponding counterterm contribution restores the satisfaction of the Yang-Baxter equation at one-loop and agreement with the exact S-matrix of [28]. Also, in the $N=4$ case the counterterm restores the group factorisation of the S-matrix under $\mathfrak{so}(4) \cong \mathfrak{su}(2)\oplus\mathfrak{su}(2)$. However, as in the reduced $AdS_5\times S^5$ theory, in the non-abelian case, $N>2$ the addition of the counterterms does not restore the validity of YBE.

The part with coefficient $N-2$ is proportional to the tree-level S-matrix and may be interpreted as being due to a shift in the coupling k by the dual Coxeter number of $H = SO(N)$, $c_H = N-2$. This is a recurring feature of these theories. In general, there are two shifts – $k\to k+\frac{c_H}{2}$ and $k\to k+c_H$ that play a rôle, hence we define the following shifted couplings,

$$\tilde{k} = k + \frac{c_H}{2}\,, \qquad \hat{k} = k + c_H\,. \tag{G.6}$$

It is useful to extract the phase factor

$$P_B(\theta,k) = 1 + \frac{i\pi}{\tilde{k}}\operatorname{cosech}\theta - \frac{\pi^2}{2\tilde{k}^2}(\operatorname{cosech}\theta)^2 + \mathcal{O}(\frac{1}{\tilde{k}^3})\,. \tag{G.7}$$

This phase factor satisfies the unitarity and crossing symmetry relations,

$$P_B(\theta,k)\ P_B(-\theta,k) = 1 + \mathcal{O}(\frac{1}{\tilde{k}^3})\,, \qquad P_B(\theta,k) = P_B(i\pi-\theta,k)\,. \tag{G.8}$$

Then the total S-matrix coefficients are given by

$$\begin{aligned}S_i = P_B\hat{S}_i\ ,\quad \hat{S}_2(\theta,k) =& 1 - \frac{\pi^2}{\hat{k}^2}\coth^2\theta + \mathcal{O}(\frac{1}{\hat{k}^3})\,, \\ \hat{S}_3(\theta,k) = \hat{S}_1(i\pi-\theta,k) =& \frac{i\pi}{\hat{k}}\coth\theta + \frac{i\pi}{2\hat{k}^2}(N-2)\theta\coth^2\theta + \mathcal{O}(\frac{1}{\hat{k}^3})\,.\end{aligned} \tag{G.9}$$

Note that while in the phase factor k enters as $\tilde{k}$, in the $\hat{S}_i$ it enters as $\hat{k}$.

G.1 $N=1$: sine-Gordon model

In the $N=1$ case the index m only takes a single value and there is only a single amplitude [61]

$$\begin{aligned}S(\theta,k) =& P_B(\theta,1)(\hat{S}_1(\theta,1) + \hat{S}_2(\theta,1) + \hat{S}_3(\theta,1)) \\ =& 1 + \frac{i\pi}{k}\Big(1+\frac{1}{2k}\Big)\operatorname{cosech}\theta - \frac{\pi^2}{2k^2}\operatorname{cosech}^2\theta + \mathcal{O}(k^{-3})\,.\end{aligned} \tag{G.10}$$

[60] This determinant contribution is present only for $N\geq 2$ (for $N=1$ or the sine-Gordon model the group H is trivial).

[61] Taking $N=1$ and summing the three determinant contributions gives $\Delta S_1(\theta,k) + \Delta S_2(\theta,k) + \Delta S_3(\theta,k) = 0$. This is expected as for $N=1$ the group H is trivial and thus there is no functional determinant contribution.

This agrees with the expansion of the exact S-matrix for the perturbative excitations of the sine-Gordon model

$$S^{(1,1)}_{SG}(\theta,\Delta(k)) = \frac{\sinh\theta + i\sin\Delta(k)}{\sinh\theta - i\sin\Delta(k)}\,, \qquad \Delta(k) = \frac{\pi}{k-\frac{1}{2}}\,, \tag{G.11}$$

where [62] for $N=1$, $\hat{S}_1(\theta,k)+\hat{S}_2(\theta,k)+\hat{S}_3(\theta,k)=1$, and thus all the information is contained in the phase factor $P_B(\theta,k)$ (G.7).

G.2 $N=2$: complex sine-Gordon model

In the $N=2$ case H is abelian. Its dual Coxeter number vanishes and the coupling k is unshifted, $\tilde{k}=\hat{k}=k$. Usually for the complex sine-Gordon model one would take the coset $G/H = SU(2)/U(1)$ rather than $G/H = SO(3)/SO(2)$. For the perturbative S-matrix the only difference amounts to a rescaling k by 2. This is a consequence of the dual Coxeter number of $SU(2)$ being twice that of $SO(3)$.

The determinant contribution is non-trivial and as H is abelian we expect the corrections to restore the satisfaction of YBE. This can be seen easily by noting that for $N=2$ the reflection coefficient in the corrected S-matrix vanishes,

$$R(\theta,k) = \hat{S}_1(\theta,k)+\hat{S}_3(\theta,k) = 0 + \mathcal{O}(k^{-3})\,. \tag{G.12}$$

If the reflection coefficient vanishes and we have a crossing symmetry then there is only one independent amplitude. The S-matrix can then be encoded in a single function

$$P_B(\theta,k)\Big(\hat{S}_2(\theta,k)+\hat{S}_3(\theta,k)\Big) = 1 + \frac{i\pi}{k}\coth\frac{\theta}{2} - \frac{\pi^2}{2k^2}\coth^2\frac{\theta}{2} + \mathcal{O}(k^{-3})\,. \tag{G.13}$$

This then agrees with the exact result derived based on assumption of exact integrability [28]. [63]

G.3 $N=4$: group factorization

For $H=SO(4)$ the field X_m transforms in a vector representation of H. As we have the isomorphism $\mathfrak{so}(4) = \mathfrak{su}(2)\oplus\mathfrak{su}(2)$, with the vector representation of $SO(4)$ equivalent to the bifundamental of $SU(2)\times SU(2)$ we can rewrite the S-matrix using the $SU(2)$ indices

$$\hat{S}^{pq}_{mn}(\theta,k) \;\sim\; \hat{S}^{c\gamma,d\delta}_{a\alpha,b\beta}(\theta,k)\,. \tag{G.14}$$

Due to the integrability of the theory this S-matrix should factorise into the tensor product of two $SU(2)$ S-matrices,

$$\hat{S}^{c\gamma,d\delta}_{a\alpha,b\beta}(\theta,k) = \hat{S}^{cd}_{ab}(\theta,k)\;\hat{S}^{\gamma\delta}_{\alpha\beta}(\theta,k)\,. \tag{G.15}$$

This is indeed the case for the S-matrix (G.9) with $N=4$

$$\begin{aligned}\hat{S}^{cd}_{ab}(\theta,k) &= \hat{s}_1(\theta,k)\delta^c_a\delta^d_b + \hat{s}_2(\theta,k)\delta^d_a\delta^c_b\,,\\ \hat{s}_1(\theta,k) &= 1 - \frac{i\pi}{2\hat{k}}\coth\theta + \frac{i\pi}{8\hat{k}^2}(5i\pi-4\theta)\coth^2\theta + \mathcal{O}(\frac{1}{\hat{k}^3})\\ \hat{s}_2(\theta,k) &= \frac{i\pi}{\hat{k}}\coth\theta - \frac{i\pi}{2\hat{k}^2}(i\pi-2\theta)\coth^2\theta + \mathcal{O}(\frac{1}{\hat{k}^3})\end{aligned} \tag{G.16}$$

These functions satisfy the crossing symmetry relations $\hat{s}_1(i\pi-\theta) = \hat{s}_1(\theta,k)+\hat{s}_2(\theta,k)$, $\hat{s}_2(i\pi-\theta) = -\hat{s}_2(\theta,k)$. The S-matrix (G.16) does not satisfy the Yang-Baxter equation [2]. Motivated by the discussion of the reduced $AdS_5\times S^5$ theory in section 6, we may consider a quantum-deformed $SU(2)$ S-matrix taking the following form (with the quantum deformation parameter $q=\exp\big(-\frac{i\pi}{\hat{k}}\big)$)

$$\begin{aligned}\mathcal{S}\,|\phi_1\phi_1\rangle &= (r_1+r_2)\,|\phi_1\phi_1\rangle\\ \mathcal{S}\,|\phi_1\phi_2\rangle &= r_1\sec\frac{\pi}{\hat{k}}\,|\phi_1\phi_2\rangle + \big(r_2 - ir_1\tan\frac{\pi}{\hat{k}}\big)\,|\phi_2\phi_1\rangle\\ \mathcal{S}\,|\phi_2\phi_1\rangle &= r_1\sec\frac{\pi}{\hat{k}}\,|\phi_2\phi_1\rangle + \big(r_2 + ir_1\tan\frac{\pi}{\hat{k}}\big)\,|\phi_1\phi_2\rangle\\ \mathcal{S}\,|\phi_2\phi_2\rangle &= (r_1+r_2)\,|\phi_2\phi_2\rangle\;\;.\end{aligned} \tag{G.17}$$

[62] In footnote 46 the shift in k for bosonic the sine-Gordon model was given as $k\to k-1$. Here k has been rescaled by 2 as we are considering a truncation of the $G/H = SO(N+1)/SO(N)$ theory. As G is abelian in this case there is no quantization of k and thus this rescaling is arbitrary.

[63] k has been rescaled by a factor of 2 relative to the one in [28], see above.

The quantum-deformed crossing symmetry relations are

$$r_1(i\pi-\theta)=\cos\frac{\pi}{\hat{k}}\big[r_1(\theta,k)+r_2(\theta,k)\big]\,,\qquad r_2(i\pi-\theta)=-\cos\frac{\pi}{\hat{k}}\big[r_2(\theta,k)-\tan^2\frac{\pi}{\hat{k}}\;r_1(\theta,k)\big]\,. \tag{G.18}$$

In analogy with the reduced $AdS_5\times S^5$ theory one may find the "closest" functions to (G.16) satisfying the crossing relations (G.18) such that the quantum-deformed S-matrix (G.17) is consistent with YBE:

$$\begin{aligned} r_1(\theta,k)=&P_0(\theta,k)\big(1-\frac{i\pi}{2\hat{k}}\coth\theta+\frac{i\pi}{8\hat{k}^2}(5i\pi\coth^2\theta-4\theta\,\mathrm{cosech}^2\theta)+\mathcal{O}(\frac{1}{\hat{k}^3})\big)\,,\\ r_2(\theta,k)=&P_0(\theta,k)\big(\frac{i\pi}{\hat{k}}\coth\theta-\frac{i\pi}{2\hat{k}^2}(i\pi\coth^2\theta-2\theta\,\mathrm{cosech}^2\theta)+\mathcal{O}(\frac{1}{\hat{k}^3})\big)\,. \end{aligned} \tag{G.19}$$

Similarly to the reduced $AdS_5\times S^5$ theory these functions agree precisely with (G.17) at tree-level with a modification of the undressed θ terms at one-loop

G.4 Symmetries

The classical bosonic $SO(N+1)/SO(N)$ theories arise as the Pohlmeyer reductions of the string theory on $\mathbb{R}_t\times S^{N+1}$, with $S^{N+1}=SO(N+2)/SO(N+1)$. Their symmetries fall into two classes, depending on $H=SO(N)$ being abelian or non-abelian.

In the case of abelian H the symmetry group is given by

$$\mathfrak{iso}(1,1)\oplus\mathfrak{h}\oplus\mathfrak{h}^{(g)}\,, \tag{G.20}$$

where the superscript (g) denotes the gauge symmetry. The fields on which the global part of the gauge symmetry has a linear action are field redefinitions of the fields on which the global H symmetry has a linear action. Therefore, the physical symmetry acting on on-shell states is

$$\mathfrak{iso}(1,1)\oplus\mathfrak{h}\,. \tag{G.21}$$

In the abelian H case the perturbative S-matrix (including the determinant corrections arising from gauge-fixing etc.) satisfies the Yang-Baxter equation.

In the case of non-abelian H the symmetry group is given by

$$\mathfrak{iso}(1,1)\oplus\mathfrak{h}^{(g)}\,, \tag{G.22}$$

i.e. there is no additional global H symmetry. The perturbative S-matrix computed following [1, 2] has a manifest symmetry given by the global part of the gauge group H. However, this S-matrix does not satisfy the Yang-Baxter equation already at the tree level.

As in the examples discussed in section 5, we may expect the physical symmetry of these theories to be given by

$$U_q(\mathfrak{iso}(1,1)\oplus\mathfrak{h})\,. \tag{G.23}$$

In the case of the abelian H this quantum deformation should have no effect as the perturbative computation agrees with integrability results and satisfies the YBE [38, 28, 2]. In the non-abelian H case (section G.3) there are quantum-deformed S-matrices (e.g., (G.17), (G.18)) closely related to the perturbative S-matrix that satisfies the Yang-Baxter equation and have a quantum-deformed H symmetry.

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